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USING MIXTURES of TRUNCATED EXPONENTIALS for SOLVING STOCHASTIC PERT NETWORKS

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Abstract

In this paper, we transform a PERT network into a mixtures of truncated exponentials Bayesian network. We use the Shenoy-Shafer architecture to propagate the MTE potentials in the resulting MTE PERT Bayes net and thus to find the marginal distribution of the project completion time. Finding the distribution of the project completion time is important because there is no closed form expression for the distribution of the maximum of two normal distributions and this fact, previously forced the researchers to make false assumptions about its distribution. In this research, we show that by approximating the maximum of two distributions using MTE's a very accurate estimation for the project completion time can be obtained.

1 Introduction

Large projects contain a series of activities that possess precedence constraints which makes project completion time difficult to manage. One of the most famous project management techniques is *Program Evaluation and Review Technique* (PERT). PERT was invented in 1958 for the POLARIS missile program by the Program Evaluation branch of the Special Projects Office of the U. S. Navy [Malcolm *et al*. 1959]. PERT networks are directed acyclic networks where the nodes represent duration of activities and the arcs represent precedence constraints. The easy applicability of PERT networks to all kind of projects made it widely used in practice. However, although a project may be represented with good accuracy using PERT networks, the accurate estimation of the project completion time is not an easy task to fulfill.

The classical solution [Malcolm *et al*., 1959] for PERT networks assumes that all activities are independent random variables, having approximate beta distributions parameterized by three parameters: mean time m, minimum (optimistic) completion time a, and maximum (pessimistic) completion time b. Using the expected duration times we compute the path that takes the longest time to finish (the critical path), hence the project completion time.

In order to involve uncertainty in the computation of project completion time and hence to improve the accuracy of the estimations, Sculli [1983] suggested to assume that all activity durations are independent, having the Gaussian distribution. This suggestion is good in the sense that it involves the uncertainty of activity durations in the computation of the project completion time. However, with this method it is also assumed that the distributions of the activity completion times are Gaussian. The completion time of an activity i is given by $C_i = \text{Max}\{C_j \mid j \in \Pi(i)\} + D_i$, where C_j denotes the completion time of activity j, D_j denotes the duration of activity j, and $\Pi(i)$ denotes the parents (immediate predecessors) of activity i. The maximum of two independent Gaussian random variables is not Gaussian, but the distribution of C_i is assumed to be Gaussian with the parameters estimated from the parameters of the parent activities. The current methods in the literature fail to recognize the true distribution of the maximum of two independent distributions and thus make false assumptions, like the maximum of two normal distributions are again normally distributed. Depending on the value of parameters this assumption can lead to large errors for the completion time of the activities which will lead to inaccurate estimates for the project completion time.

Motivated by this problem in the literature, Cinicioglu and Shenoy [2006] provided a new method which aims to approximate the true distribution of the project completion time by eliminating the false assumptions for the distribution of the maximum of two Gaussians. With this method, a PERT network is transformed into a mixtures of Gaussians Bayesian network and then Lauritzen-Jensen algorithm is used to make inferences in the resulting MoG Bayesian network. Mixtures of Gaussians (MoG) hybrid Bayesian networks [Lauritzen, 1992] are Bayesian networks with a mix of discrete and continuous variables. In MoG Bayesian networks the discrete variables cannot have continuous parents, and all continuous variables have the so-called conditional linear Gaussian distributions.

Representation of a PERT network as a MoG Bayesian network is beneficial in the sense that it eliminates the false assumption made in the literature which assumes that the maximum of two normally distributed independent random variables is again normally distributed. However, the transformation process of a PERT network into a MoG Bayesian network is cumbersome because of the restricted nature of MoG Bayesian networks. The inability of discrete variables to have continuous parents and the enforcement for continuous variables to possess conditional linear Gaussian distributions makes the transformation process of a PERT network into a MoG Bayes net too complex for practical use.

For that reason, in this research we work on a different method, an alternative to MoG Bayesian networks, which overcomes the difficulties involved in solving stochastic PERT networks using MoG's, but still possess the advantages involved in it. The alternative we suggest in this paper for solving stochastic PERT networks with MoGs, is to solve them using mixtures of truncated exponentials (MTE). We proceed as follows: First we transform a PERT network into a PERT Bayes net, so we can model the dependencies between activity durations. Next, we transform the PERT Bayes net into a MTE network by approximating the activity durations using MTE's. Finally using the Shenoy-Shafer architecture we propagate the MTE potentials and find the marginal distribution of the project completion time. To evaluate our method we compare the mean and variance of the marginal distribution of the project completion time with the exact analytic results using Clark's method [1961] and the shape of our distribution with the actual distribution calculated by brute force using order statistics.

2 Representation of a PERT network as a Bayesian network

In order to demonstrate our method of solving stochastic PERT networks using mixtures of truncated exponentials we will use a simple example of a PERT network and compute the marginal distribution of the project completion time. Consider the PERT network given in Figure 1 below. This network represents a project with the activities A_1, A_2 and A_3. S stands for the project start time and E stands for the project completion time. We assume that the project start time is zero. The precedence constraints, represented by arcs, are as follows: The activities A_1 and A_2 do not have any predecessors. The activity A_3 can only be started after A_1 is completed.

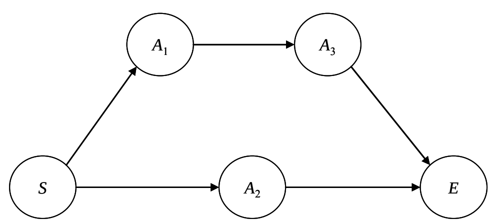

Figure 1. An example of a stochastic PERT network with three activities

The distributions of activity durations are known, and we are informed that the activity durations A_1 and A_3 are positively correlated. Following the method described in Jenzarli[1995] this PERT network will be transformed into a PERT Bayesian network in four basic steps, allowing us to model the dependencies between the activity durations.

Let D_i and C_i denote the duration and the completion time of the activity i, respectively. As the first step of the transformation process, the activity durations are replaced with activity completion times. Next, activity durations will be added with an arrow from D_i to C_i, so that each activity will be represented by two nodes, its duration D_i and its completion time C_i. As the next step, notice that the completion times of the activities which do not have any predecessors will be the same as their durations. Hence, these activities A_1 and A_2 will be represented just by their durations, as D_1 and D_2. Remember that we are informed that the activities D_1 and D_3 are positively correlated. As the last step of the transformation process, the dependency between these activity durations will be depicted by adding an arrow from D_1 to D_3. We assume that the project start time is zero with probability 1 and each activity will be started as soon as all the preceding activities are completed. Accordingly, E represents the completion time of the project, which is the $Max\{D_2, C_3\}$. The resulting PERT Bayes net is given in Figure 2 below. Notice that the deterministic variables, C_3 and E, are depicted as double bordered ovals. The next section describes mixtures of truncated exponentials.

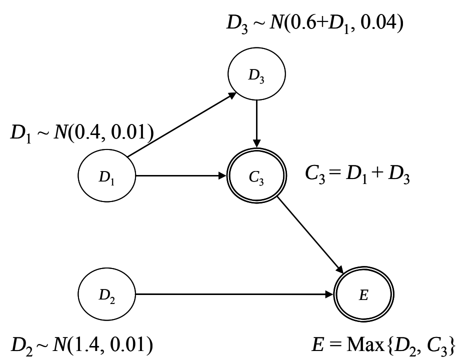

Figure 2: An example of a PERT Bayesian network

3 Mixtures of Truncated Exponentials

MTE's are an alternative to discretization and Monte Carlo methods for solving hybrid Bayesian networks [Moral *et al.* , 2001; Rumi, 2003]. MTE potentials can be used for inference in hybrid Bayesian networks that do not fit the restrictive assumptions of the conditional linear Gaussian (CLG) model, such as networks containing discrete nodes with continuous parents.

A mixture of truncated exponential (MTE) [Moral *et al.* , 2001; Rumi, 2003] has the following definition.

Let X be a mixed n-dimensional random variable. Let $Y = (Y_1, \ldots, Y_d)$ and $Z = (Z_1, \ldots, Z_c)$ be the discrete and continuous parts of X, respectively, with $c + d = n$. A function $\phi: \Omega_X \mapsto R^+$ is an MTE potential if one of the next two conditions holds:

The potential ϕ can be written as

$$\phi(x) = \phi(y, z) = a_0^y + \sum_{i=1}^{m} a_i^y \exp(\sum_{j=1}^{c} b_j^y z_j) \qquad (3.1)$$

where a^y_0, a^y_i and b^y_j are real numbers for all $i = 1, \ldots, m$, $j = 1, \ldots, c$, $y \in \Omega_Y$ and $z \in \Omega_Z$.

There is a partition $\Omega_1, \ldots, \Omega_k$ of Ω_X verifying that the domain of continuous variables, Ω_Z, is divided into hypercubes, the domain of the discrete variables, Ω_Y, is divided into arbitrary sets, and such that ϕ is defined as $\phi(x) = \phi_i(x)$ if $x \in \Omega_i$, where each ϕ_i, $i = 1, \ldots, k$ can be written in the form of equation (3.1)

In the definition above, k is the number of pieces and m is the number of exponential terms in each piece of the MTE potential.

The nice thing about MTE's is that any probability density function can be approximated by an MTE potential, which can always be marginalized in closed form. Consider a normally distributed random variable X with mean μ and variance $\sigma^2 > 0$. The PDF for the normal distribution is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-1/2\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

A general formulation for a 2-piece, 3-term unnormalized MTE potential which approximates the normal PDF is as follows [Cobb and Shenoy, 2006a].

$$\psi'(x) = \begin{cases} \sigma^{-1}(-0.010564 + 197.055720\exp\{2.2568434(\frac{x-\mu}{\sigma})\} \\ -461.439251\exp\{2.3434117(\frac{x-\mu}{\sigma})\} \\ +264.793037\exp\{2.4043270(\frac{x-\mu}{\sigma})\}) \quad if \quad \mu-3\sigma \le x < \mu \\ \sigma^{-1}(-0.010564 + 197.055720\exp\{-2.2568434(\frac{x-\mu}{\sigma})\} \\ -461.439251\exp\{-2.3434117(\frac{x-\mu}{\sigma})\} \\ +264.793037\exp\{-2.4043270(\frac{x-\mu}{\sigma})\}) \quad if \quad \mu-3\sigma \le x < \mu \\ 0 \quad otherwise \end{cases} \tag{3.2}$$

In the following sections the PERT network example will be transformed into a MTE PERT Bayesian network and solved using the Shenoy-Shafer architecture. The operations necessary to carry out propagation in MTE networks using the Shenoy-Shafer architecture are described in the following, subsection 3.1.

3.1 Operations in MTE Networks

This section describes the operations of restriction, combination, marginalization, normalization, operations with linear deterministic equations and finding the maximum of two distributions using MTE's. These operations are necessary to carry out propagation in our MTE network example. The class of MTE potentials is closed under these operations which allows us to use the Shenoy-Shafer architecture [Shenoy and Shafer, 1990] to propagate the MTE potentials in the network. The definitions of restriction, combination, marginalization and normalization are described in Moral *et al*. [2001]. The operations with linear deterministic variables in MTE networks are described in Cobb and Shenoy[2005]. The operations for finding the maximum of two distributions using MTE's are first described here.

3.1.1 Restriction

Restriction is the operation of entering evidence during the propagation. In restriction, known variables are substituted with their values.

Let ϕ be an MTE potential for $X = Y \cup Z$. Suppose we receive the evidence for a set of variables $X' = Y' \cup Z' \subseteq X$, s.t. its values $x^{\downarrow\Omega X'}$ are as follows: $x' = (y', z')$. After receiving the evidence the values of the variables are known. Accordingly, the potential ϕ should be updated. The new potential defined on $\Omega_{X\backslash X'}$ is as follows:

$$\phi^{R(X' = x')}(w) = \phi^{R(Y' = y',\, Z' = z')}(w) = \phi(x) \qquad (3.3)$$

for all $w \in \Omega_{X\backslash X'}$ such that $x \in \Omega_X$, $x^{\downarrow\Omega X\backslash X'} = w$ and $x^{\downarrow\Omega X'} = x'$. In this definition each occurrence of X' in ϕ is replaced with x'. An example for restriction is provided in section 6.

3.1.2 Combination

MTE potentials are combined by pointwise multiplication. Let ϕ_1 and ϕ_2 be the MTE potentials for $X_1 = Y_1 \cup Z_1$ and $X_2 = Y_2 \cup Z_2$. The combination of ϕ_1 and ϕ_2 is a new MTE potential for $X = X_1 \cup X_2$ defined as follows:

$$\phi(x) = \phi_1(x^{\downarrow X_1})\, \phi_2(x^{\downarrow X_2}) \text{ for all } x \in \Omega_x \qquad (3.4)$$

3.1.3 Marginalization

MTE potentials are marginalized by summing over discrete variables and integrating over continuous variables. Let ϕ be an MTE potential for $X = Y \cup Z$. The MTE potentials are closed under marginalization, so the marginal of ϕ for the set of variables $X' = Y' \cup Z' \subseteq X$ is a MTE potential which is computed as follows:

$$\phi^{\downarrow X'}(y', z') = \sum_{y \in \Omega_{Y\backslash Y'}} \Big(\int_{\Omega_{Z\backslash Z'}} \phi(y, z)\, dz'' \Big) \qquad (3.5)$$

where $z = (z', z'')$, and $(y', z') \in \Omega_{X''}$. The variables can be marginalized in any sequence, discrete before continuous or continuous before discrete as shown in Formula 3.5.

In the process of marginalization, when the limits of integration include linear functions, then we may end up with linear terms in the remaining variables. These linear terms can be replaced with an MTE approximation so that the result of the marginalization is again an MTE potential. For a linear term x defined over the domain $[x_{min}, x_{max}]$, we replace x with

$$x_{\min} + (x_{\max} - x_{\min})(0.5 * (-13.5070292 + 13.5070292\,Exp[\frac{0.0726981(x - x_{\min})}{(x_{\max} - x_{\min})}] \quad (3.6)$$
$$+ 0.5 * (13.5070364 - 13.5070364 Exp[\frac{-(0.0754406(x - x_{\min})}{(x_{\max} - x_{\min})}]$$

The replacement of the linear terms ensures that MTE potentials are closed under marginalization.

3.1.4 Normalization

Let $X = Y \cup Z$ be a set of variables where Y is a discrete and Z is a continuous variable. Let ϕ' be the MTE potential for X. Normalization constant for K is calculated as follows:

$$K = \sum_{y \in \Omega_Y} (\int_{\Omega_Z} \phi'(y, z) dz) \qquad (3.7)$$

If join trees are initialized with normalized potentials the normalization constant equals to one when no evidence is observed.

3.1.5 Linear Deterministic Equations

If the variable being deleted is contained in a linear deterministic equation in the network, then the marginalization operation is different. If it is the case, then we solve the equation for the variable being deleted and then substitute this solution in the updated potentials in the network.

Let ψ denote the distribution of $Y|x \sim f_{Y|X}$ and let ζ denote the equation $Z = X + Y$. Suppose we want to delete the variable Y from the network. By solving the equation for Y and substituting the solution in $f_{Y|x}$ we can remove Y out of the combination and hence find the distribution of $Z|x$. The details are as follows:

$(\zeta \otimes \psi)^{-Y} = ([Z = X + Y] \otimes f_{Y|X}(y))^{-Y} = ([Y = Z - X] \otimes f_{Y|X}(y))^{-Y} = f_{Y|x}(z - x)$

3.1.6 Maximum of Two Distributions

Finding the distribution of the maximum of two or more distributions has been the interest of many communities of researchers. Especially in the domains of project management, this problem occupies an important place since the completion time of an activity is the sum of its duration and the maximum between the completion times of its immediate predecessors. For this reason, it can be concluded that an accurate estimation of the project completion time is very much affected by an accurate estimation of the activity completion times.

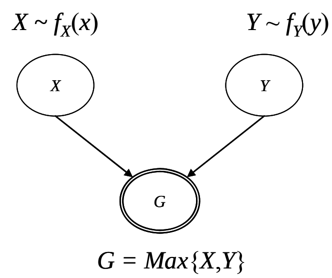

Figure 3. Maximum of two distributions

The marginal probability density function of the maximum of two distributions can be computed by brute force using order statistics. Consider the small BN given in Figure 3. X and Y are continuous variables which have density functions $f_X(x)$ and $f_Y(y)$, respectively. G is a deterministic variable which is distributed as $G = Max\{X, Y\}$. Let F_G denote the cumulative distribution function (CDF) of G, F_X denote the CDF of X and F_Y denote the CDF of Y. Then, $F_G(g) = F_X(g)F_Y(g)$. Therefore, the probability density function of G is given by $f_G(g) = (d/dg)F_G(g) = f_X(g)\ F_Y(g)+ F_X(g)\ f_Y(g)$, where f_X and f_Y are the PDFs of X and Y, respectively. Since there is no closed form expression for the CDF of a normal distribution, there is no closed form expression for $f_G(g)$ when X and Y are normally distributed. Since MTE potentials are closed under integration both $F_X(g)$ and $F_Y(g)$ can be expressed as MTE potentials. And since MTE potentials are closed under multiplication and addition $f_G(g)$ can also be expressed as MTE potentials. Then, by using the MTE approximations of X and Y, we can obtain an MTE approximation for the distribution of $f_G(g)$.

The next section describes the transformation of our PERT Bayes net example into a MTE PERT Bayesian network.

4 Transformation of a PERT Bayesian network into a MTE PERT Bayesian network

The primary objective of this study is to compute the completion time of the project without setting any assumptions for activity distributions. This objective will be materialized by approximating the activity durations using mixtures of truncated exponentials and propagating the resulting mixtures of truncated exponentials network using the Shenoy-Shafer architecture.

Consider the PERT Bayes net given in Figure 2. Notice that it is not a MTE Bayesian network since the activity durations D_1, D_2, and D_3 are all normally distributed. In order to transform this PERT Bayes net into a MTE Bayesian network all of these activities will be approximated using MTE's. The MTE approximation of D_1 overlaid on the actual normal distribution is given in Figure 4 below.

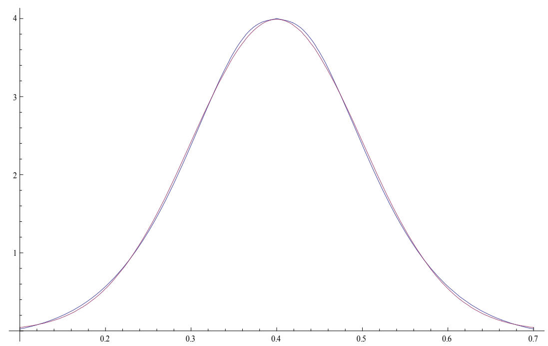

Figure 4. The actual distribution of D_1 overlaid on its MTE approximation

The probability distribution for D_3 is defined as $D_3|d_1 \sim N(0.6+d_1, 0.04)$. The plot for the MTE approximation for D_3 is given in Figure 5 below.

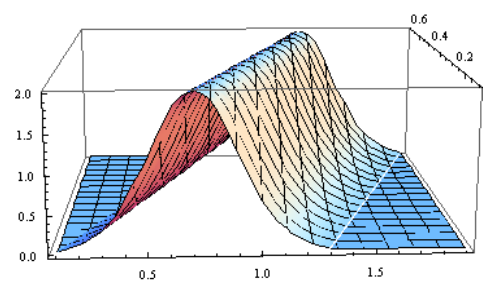

Figure 5. MTE approximation for $D_3|d_1$

5 Fusion Algorithm

The fusion algorithm, first described by Cannings *et al.* [1978], is used to compute the marginal for a variable using local computation [Shenoy, 1992]. Shenoy [1997] described the fusion algorithm as a guide to construct join trees where Shenoy-Shafer architecture will be used to compute the marginals of the variables. The basic idea of the fusion algorithm is to delete all the variables in the network successively, until we end up with the marginal distribution of the variable of interest.

In this research, we are interested in computing the marginal distribution of the project completion time. Hence, using fusion algorithm, the variables in the MTE PERT Bayes net will be deleted successively, until we end up with the marginal distribution of the project completion time, *F*. Though different deletion sequences may lead to different computational efforts, the outcome of the network does not get affected with the deletion sequence used. In this example, we will use the deletion sequence D_3, D_1, (D_2, C_3) in order to find the marginal distribution of the project completion time. Figure 6 illustrates the construction of the join tree for the PERT example.

The details of the messages necessary to compute the marginal distribution of the project completion time are as follows:

Fusion with respect to D_3:

Fusion w.r.t. D_3, refers to removing the variable D_3 from the network. This will be done first by combining all the potentials that contain D_3 and next by removing D_3 out of the combination by marginalizing the combination down to the remaining variables. Let $f_{D_3|d_1}$ denote the distribution of $D_3|d_1$. Let χ_3 denote the equation for $C_3 = D_1 + D_3$. By solving the equation for D_3 and substituting D_3 in $f_{D_3|d_1}$ we can find the distribution of $C_3|d_1$. The details are as follows:

$C_3 = D_1 + D_3$

$D_3 = C_3 - D_1$

$f_{C_3|d_1}(c_3) = f_{D_3|d_1}(c_3 - d_1)$

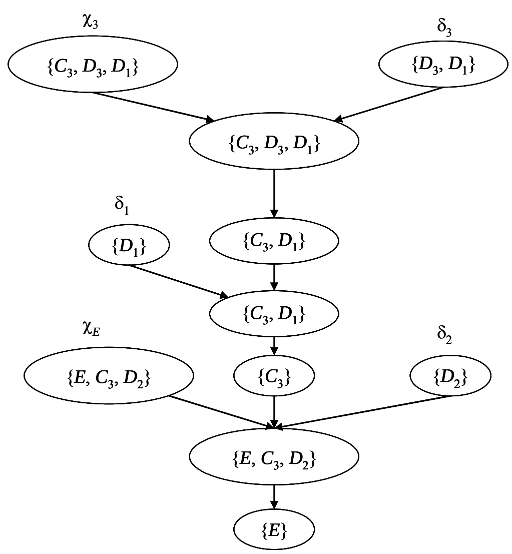

Figure 6. Creation of the binary join tree using the fusion algorithm.

Fusion with respect to D_1:

The variables whose domains contain D_1, (D_1 itself and $C_3|d_1$), are both continuous variables, so deleting D_1 from the network involves finding the joint $f_{C_3,\,D_1}(c_3,\,d_1)$ and integrating this combination over the domain of D_1. The details are as follows:

$f_{C_3,\,D_1}(c_3,\,d_1) = f_{C_3|d_1}(c_3)\, f_{D_1}(d_1)$

$(f_{C_3,\,D_1}(c_3,\,d_1))^{\downarrow C_3} = \int f_{C_3,D_1}(c_3,\,d_1)\, dd_1 = f_{C_3}(c_3)$

The expected value and variance for the marginal of C_3 are calculated as 1.4 and 0.0786. These answers are comparable with results from multivariate normal theory, which gives an expected value and variance of 1.4 and 0.08.

The next step is to find the marginal distribution of $E = Max\{C_3, D_2\}$ which requires the variables, C_3 and D_2, to be deleted at the same time.

Figure 7 represents the current state of our network after the variables D_3 and D_1 are removed from the network. As the next and final step, we have to find the project completion time $E = Max\{C_3, D_2\}$ which requires the variables C_3 and D_2 to be deleted at the same time.

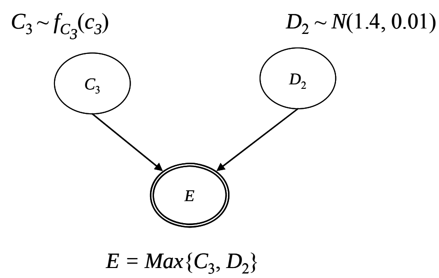

Figure 7. The conditional distribution of E after D_3 and D_1 are deleted from the network

As explained in subsection 3.1.6, the probability density function of F_E is given by $f_E(e) = (d/de)F_E(e) = f_{C_3}(e)\, F_{D_2}(e) + F_{C_3}(e)\, f_{D_2}(e)$, where f_{C_3} and f_{D_2} are the PDFs of C_3 and D_2, respectively. In sections 4 and 5 the PDF's of D_2 and C_3 are approximated using MTE's. As the next step of our analysis, we calculate the CDF's of both D_2 and C_3 which we later use for the calculation of the marginal distribution of the project completion time, $f_E(e)$. The plot of the MTE approximation for the CDF of D_2 is illustrated in Figure 8 below.

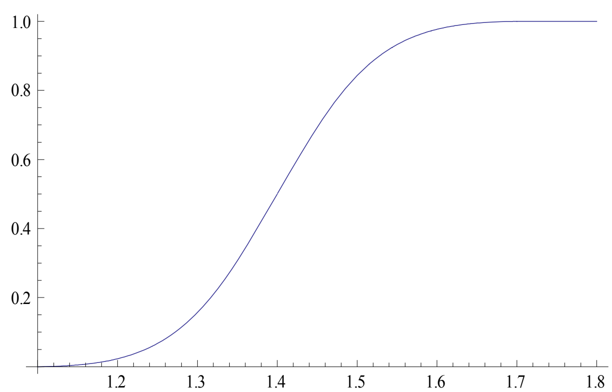

Figure 8. MTE Approximation for $F_{D_2}(e)$

The MTE approximation of $f_E(e)$ overlaid on the actual distribution is given in Figure 9 below.

By comparing the means and variances of the approximation with the exact analytic results calculated with Clark's method [1961], we can evaluate the goodness of our approximation for the marginal distribution of the project completion time, $f_E(e)$. Accordingly, using our method described in this paper the mean and the variance of the marginal distribution of E is calculated as 1.51883 and 0.0300638, respectively. Comparing it to 1.51968 and 0.0306761 given by the exact analytic results, the approximation can be considered as quite successful.

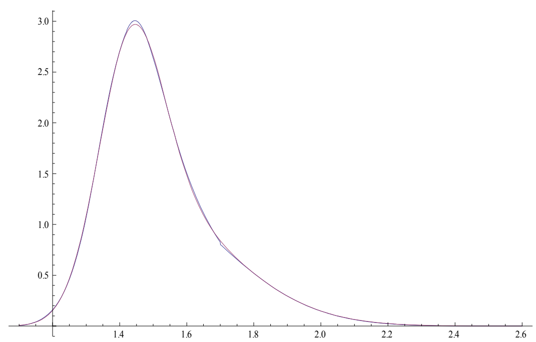

Figure 9. Approximation of $f_E(e)$overlaid on the actual distribution

After normalization, when the limits of integration include linear terms, then we may end up with linear terms in the remaining variables as it is the case with the approximation of C_3 and of the CDF of D_2. These linear terms can be approximated again using MTE potentials, which ensures that the result is again an MTE approximation and MTE's are closed under marginalization. However, replacing the linear terms with the MTE potentials causes bad accuracy in our approximations.

6 Entering Evidence in a MTE PERT Network

In this research MTE PERT Bayes nets are described as an alternative method to solve stochastic PERT networks with which we can compute the marginal distribution of the project completion time without setting any false assumptions for the activity completion times. In this context, it is natural to question our methods described in this research and ask for the advantage obtained by using the methods described, instead of using straight forward simulation methods that are already handy.

With simulation methods the activity durations can be represented realistically. As it is the case with our methods, the activity durations can have any type of distribution and one can also represent the correlation between the activity durations. However, with straight-forward Monte Carlo simulation methods we can not include the observations of continuous variables and update our inferences accordingly. By transforming the PERT network into a MTE Bayesian network and solving it using the Shenoy-Shafer architecture we can update our network, once evidence is observed, and find the posterior distributions of the activities which in turn will result in more accurate estimates for the project completion time.

Consider the PERT Bayesian network given in Figure 10. This is a PERT Bayes net with four activities A_1, A_2, A_3 and A_4. Notice that the activities are depicted by their durations, as D. Suppose we know that the activities A_1 and A_2 will be performed by the same contractor. The quality of the work done by this contractor is distributed as $f_Q(q)$. The quality of the work performed by the contractor effects the duration of the activities A_1 and A_2 such that with higher quality it will take less time to complete these activities. In addition to these, we also have the information that the same contractor performs another activity similar to ours within the firm. This activity A_4 is outside of our project but we included it in our network in Figure 10 anyway since it will effect our later conclusions. As you can see in Figure 10 the duration of activity A_4 also depends on the quality of the contractor's job.

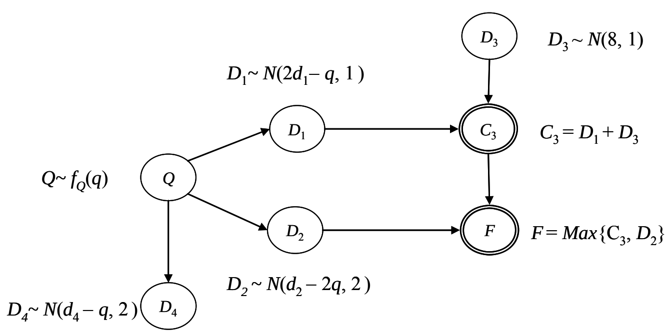

Figure 10. Representation of the example as a PERT Bayesian network

The example described above can be solved using the means of simulation methods as well as with the methods represented throughout this research. However, suppose we observe that the duration of activity A_4 lasted 10 days to complete. Hence we have the evidence e_{D_4} = 10. With the methods described in this dissertation this evidence can be incorporated in the network and the estimates for the durations can be updated accordingly, which is not possible using the straight forward simulation. With our method we can find the posterior distribution of Q after receiving the evidence e_{D_4} which in turn will change the estimates for the distributions of A_1 and A_2 and consequently the estimate for the project completion time. Including the observations in the network and updating the distributions accordingly will improve the quality of the inference. The PERT BN after receiving the evidence e_{D_4} is represented in Figure 11 below.

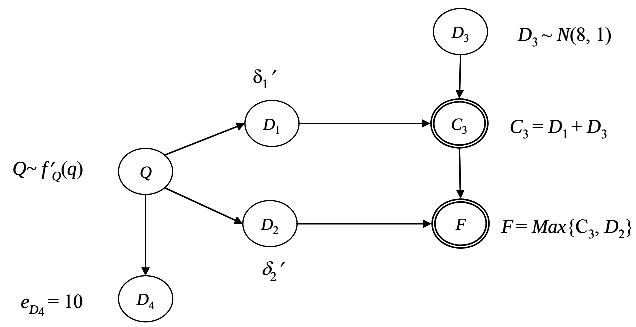

Figure 11. The PERT Bayesian network after receiving the evidence e_{D_4}

7 Summary and Conclusions

Mixtures of truncated exponentials are an alternative tool to mixtures of Gaussians (MoG) to make inferences in stochastic PERT networks. Both MoG's and also MTE's are able to find accurate estimations for the maximum of two distributions and hence for the project completion time. However, the inference process using MTE PERT networks, compared to MoG's, is much more straightforward in the sense that the MTE PERT networks do not force restrictive settings like, the inability of discrete variables to have continuous parents as it is the case with MoG networks. This fact makes the use MTE PERT Bayes nets better suited for practical use.

Comparing our method to straight forward simulation on the other hand, the MTE PERT Bayesian networks possess the advantage that the observations can be integrated to the inference process. Once evidence is observed we can update our network accordingly and find the posterior distributions of the activities and thus obtain a more accurate estimation for the project completion time.

The drawback with our method is on the other hand, that the number of exponential terms increases rapidly as the fusion algorithm is applied which in turn makes the inference process more difficult to apply. Additionally, in the process of marginalization, when the limits of integration include linear functions, we may end up with linear terms in the remaining variables. These linear terms can be approximated using an MTE approximation and it can be ensured that the result is again an MTE potential. However, replacing the linear terms with the MTE potentials causes bad accuracy in our approximations.

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