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# TRANSACTIONS OF THE TWENTY-FOURTH CONFERENCE OF ARMY MATHEMATICIANS

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THE CHIEF OF RESEARCH, DEVELOPMENT AND ACQUISITION

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# TRANSACTIONS OF THE TWENTY-FOURTH CONFERENCE OF ARMY MATHEMATICIANS

Sponsored by the Army Mathematics Steering Committee

Hosts

U. S. Army Foreign Science and Technology Center
with the
School of Engineering and Applied Science
University of Virginia
Charlottesville, Virginia

31 May and 1-2 June 1978



U. S. Army Research Office
P. O. Box 12211
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# FOREWORD

The theme of the Twenty-fourth Conference of Army Mathematicians was "stochastic processes". Four of the six invited speakers, listed below, spoke on topics related to this theme. In recent years there has been a shift of interest from the deterministic descriptive processes to the stochastic processes. This has been brought about by the need to explain many of the phenomena arising in such fields as physics, engineering, biology, and medicine. The complexities and uncertainties that appear in these fields have forced mathematicians to make frequent use of probabilistic concepts. Army scientists are having to deal with stochastic equations, principally those associated with ordinary and partial differential equations. These are concerned with such phenomena as wave propagation, turbulence and diffusion theory.

| Speaker and Institution | Area of Talk |
|---|---|
| Professor E. J. McShane<br>University of Virginia | Choosing a Mathematical Model for a System Affected by Noise |
| Professor R. E. Kalman<br>University of Florida | Nonlinear Realization Theory |
| Professor Y. K. Lin<br>University of Illinois | Stochastic Theory of Rotor Blade Dynamics |
| Professor Roger Brockett<br>Harvard University | Optimal Multilinear Estimators |
| Professor Ronald DiPerna<br>Mathematics Research Center<br>University of Wisconsin-Madison | Hyperbolic Conservation Laws |
| Professor Eugene Wong<br>University of California-Berkeley | A Martingale Theory of Random Fields |

The Twenty-fourth Conference of Army Mathematicians was held 31 May - 2 June 1978 at Charlottesville, Virginia. The U. S. Army Foreign Science and Technology Center (AFSATC), together with the School of Engineering and Applied Sciences of the University of Virginia, served as its hosts. Colonel Anthony P. Simkus, Commanding Officer of the US Army Research Office, played a key role in obtaining the hosts for this meeting. This fact is borne out by the following quotation from a letter by Colonel Claire J. Reeder, Commanding Officer of AFSATC. "I was pleased to receive the proposal by your office to hold the

24th Conference of Army Mathematicians in Charlottesville. As another Army Organization with a scientific and technical mission, I welcome such opportunities to interact with the Army research community. In this case the University of Virginia will be cooperating with us as joint host for the meeting and will provide the conference facilities."

This conference is part of a continuing program of Army-wide symposia held under the auspices of the Army Mathematics Steering Committee (AMSC) to promote better communication among Army scientists. In order that this mission be accomplished, a large number of individuals must expend a great deal of effort. It is not possible to single out all the persons involved in making the 1978 conference such a scientific success, but members of the AMSC would like to recognize a few of these individuals as well as certain organizations. First of all they would like to express their gratitude to the University of Virginia and the AFSATC for providing the necessary facilities and the cordial atmosphere for this conference. Special recognition is due the outstanding arrangements made possible by the two chairpersons on Local Arrangements. Mrs. Betty Jane Pruffer who handled, without a hitch, the administrative details and Mr. Kent Schlussel who handled in a similar matter the technical problems. Finally, the members of the AMSC would like to commend both the invited speakers and the authors of contributed papers for their excellent presentations and the valuable contributions of their papers to the field of science.

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*This table of contents contains only the papers that are published in this technical manual. For a list of all papers presented at the Twenty-fourth Conference of Army Mathematicians, see the Program of the meeting.

TITLE PAGE

A SOLUTION FOR NON-COOPERATIVE GAMES

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ABSTRACT. In this paper we study solutions of strict non-cooperative games t are played just once. The players are not allowed to communicate with each er. The main ingredient of our theory is the concept of rationalizing a set strategies for each player of a game. We state an axiom based on this con- t that every solution of a non-cooperative game is required to satisfy. ong Nash solvability is shown to be a sufficient condition for the rational- ng set to exist, but it is not necessary. Also, Nash solvability is neither essary nor sufficient for the existence of the rationalizing set of a game. a game with no solution (in our sense), a player is assumed to recourse to standard of behavior". Some standards of behavior are examined and discussed.

I. INTRODUCTION. In this paper, we study solutions of non-cooperative ies. In a non-cooperative game, absolutely no preplay communication is allowed ween the players. The theory of non-cooperative games, in contrast with perative games, is based on the absence of coalitions in that it is assumed it each participant acts independently without collaboration or communication h any of the others. Since in repeated plays of a game it is possible for tyers to "communicate" or signal via their choice patterns on previous plays† shall avoid this feature of a non-cooperative game by only considering games at are played just once. Our objective is to study strict non-cooperative nes and although this may be a severe restriction on the class of realistic nes, like Luce and Raiffa [6, pp. 105], we feel that

> "...the realistic cases actually lie in the hiatus between strict non-cooperation and full cooperation but that one should first attack these polar extremes."

sides, in many of the games that arise in the military and political contexts, e players often have a single-play orientation.

Except for this difference, we make the usual assumptions of rationality d complete information, i.e., all players are "rational"‡ and each player has mplete information of this fact and of his own and other players' utility nction.

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ee Luce and Raiffa [6, pp. 97-102] for a discussion of the temporal repetition f the prisoner's dilemma.

lere we mean in the usual von Neumann and Morgenstern sense. Later in Section II, we will look at this assumption more critically and study its implications.

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onsored by the United States Army under Contract No. DAAG29-75-C-0024 and by e National Science Foundation under Grant No. MCS75-17385 A01.

II. FORMAL DEFINITIONS AND TERMINOLOGY. In this section we will define the basic concepts in the non-cooperative theory. The non-cooperative idea will be implicit, rather than explicit, below.

An *n-person game* is a set of $n$ *players* denoted by $N = \{1,\ldots,n\}$, each with an associated finite set of *pure strategies*; and corresponding to each player, $i$, a *von Neumann-Morgenstern utility function* $u_i$, which maps the set of all n-tuples of pure strategies into real numbers. By the term n-tuple, we mean a set of $n$ items with each item associated with a different player. A *mixed strategy* of player $i$ will be a probability distribution on his set of pure strategies. We write $s^i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ with $c_{i\alpha} \geq 0$ and $\sum_\alpha c_{i\alpha} = 1$ to to represent such a mixed strategy, where the $\pi_{i\alpha}$'s are the pure strategies of player $i$. The von Neumann-Morgenstern utility function $u_i$ used in the definition of a finite game above has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player (n-linear). This extension we will also denote by $u_i$, writing $u_i(s^1,s^2,\ldots,s^n)$. I.e.,

$$u_i(s^1,s^2,\ldots,s^n) = \sum_{\alpha_1} \cdots \sum_{\alpha_n} c_{1\alpha_1}\cdots c_{n\alpha_n} u_i(\pi_{1\alpha_1},\ldots,\pi_{n\alpha_n}) \ .$$

We shall use the symbols $i,j,k$ for players and $\alpha,\beta,\gamma$ to indicate various pure strategies of a player. The symbols $s^i,t^i,r^i$ will indicate mixed strategies; $\pi_{i\alpha}$ will denote the $i^{th}$ player's $\alpha^{th}$ pure strategy, etc. We shall write $\bar{s},\bar{t}$ to denote an n-tuple of mixed strategies. For convenience we shall use the substitution notation $(\bar{s};t_i)$ to denote $(s^1,\ldots,s^{i-1},t^i,s^{i+1},\ldots,s^n)$ where $\bar{s} = (s^1,\ldots,s^n)$.

An n-tuple $\bar{s}$ is a *Nash equilibrium point* if and only if for every $i$

$$u_i(\bar{s}) = \max_{\text{all } t^i\text{'s}} [u_i(\bar{s};t^i)] \ .$$

Thus an equilibrium point is an n-tuple $\bar{s}$ such that each player's mixed strategy maximizes his payoff if the strategies of the others are held fixed. In an extremely elegant proof, Nash [8] has shown that every non-cooperative game with finite sets of pure strategies has an equilibrium point. A strategy $s^i$ is player $i$'s *equilibrium strategy* if the n-tuple $(\bar{t};s^i)$ is an equilibrium point for some n-tuple $\bar{t}$.

A strategy $r^i$ is player $i$'s *maximin* strategy if and only if for all n-tuples $\bar{s}$,

$$u_i(\bar{s};r^i) \geq \max_{\text{all } s^i\text{'s}} \ \min_{\text{all } s^1,\ldots,s^{i-1},s^{i+1},\ldots,s^n} [u_i(s^1,\ldots,s^n)] \ .$$

quantity on the right side of the above inequality is called player $i$'s imin value and denoted by $v_i^m$.

For 2-person games only, a strategy $t^i$ is player $i$'s minimax strategy and only if for all player $j$'s strategies, $s^j$, $j \neq i$

$$u_j(t^i, s^j) \leq \min_{\text{all } s^i\text{'s}} \max_{\text{all } s^j\text{'s}} [u_j(s^i, s^j)] .$$

We say that a mixed strategy $s^i$ uses a pure strategy $\pi_{i\alpha}$ if $= \sum_\beta c_{i\beta}\pi_{i\beta}$ and $c_{i\alpha} > 0$. If $\bar{s} = (s^1, \ldots, s^n)$ and $s^i$ uses $\pi_{i\alpha}$, we also that $\bar{s}$ uses $\pi_{i\alpha}$. Let $s^i$ and $r^i$ be two distinct mixed strategies player $i$. We say $s^i$ strongly dominates $r^i$ if $u_i(\bar{t}; s^i) > u_i(\bar{t}; r^i)$ for ry $\bar{t}$. This amounts to saying that $s^i$ gives player $i$ a higher payoff n $r^i$ no matter what the strategies of the other players are. To see ther a strategy $s^i$ strongly dominates $r^i$, it suffices to consider only e strategies for the other players because of the n-linearity of $u_i$. Also, say $s^i$ weakly dominates $r^i$ if $u_i(\bar{t}; s^i) \geq u_i(\bar{t}; r^i)$ for all $\bar{t}$ and ict inequality holds for at least one $\bar{t}$.

Based on the concept of an equilibrium point, Nash defined several lutions" of non-cooperative games. A game is said to be Nash solvable if set $S$ of equilibrium points satisfies the condition

$$(\bar{t}; r^i) \in S \text{ and } \bar{s} \in S \Rightarrow (\bar{s}; r^i) \in S . \tag{2.1}$$

s is called the interchangeability condition. The Nash solution of a Nash vable game is its set $S$ of equilibrium points. A game is strongly Nash vable if it has a Nash solution, $S$, such that for all $i$'s

$$\bar{s} \in S \text{ and } u_i(\bar{s}; r^i) = u_i(\bar{s}) \Rightarrow (\bar{s}; r^i) \in S$$

then $S$ is called a strong Nash solution. If $S$ is a subset of the set of ilibrium points of a game and satisfies condition (2.1); and if $S$ is maxi- relative to this property, then we call $S$ a Nash subsolution. Let $S$ be set of all equilibrium points of a game. Define

$$v_i^+ = \max_{\bar{s} \in S} [u_i(\bar{s})], \quad v_i^- = \min_{\bar{s} \in S} [u_i(\bar{s})] .$$

$v_i^+ = v_i^-$, we write $v_i = v_i^+ = v_i^-$. $v_i^+$ is called the Nash upper value to yer $i$ of the game; $v_i^-$ the Nash lower value; and $v_i$ the Nash value, if exists.

Note that a non-cooperative game does not always have a Nash solution, but n it does, the Nash solution is unique. Strong Nash solutions are Nash solutions with special properties. Nash subsolutions always exist and have many of the properties of Nash solutions, but lack uniqueness. A Nash subsolution, when unique, is a Nash solution.

Apart from these "solutions", Luce and Raiffa [6, Ch. 5] have defined "solution in the strict sense", "solution in the weak sense" and "solution in the complete weak sense". For reasons of space, we do not repeat these definitions here.

A natural question that arises is: In what sense are these concepts, solutions of non-cooperative games? I.e., what constitutes a solution of a non-cooperative game? These questions are discussed in the subsequent sections.

III. SOLUTIONS OF NON-COOPERATIVE GAMES. What do we mean by a solution of a non-cooperative game? Let $\Gamma$ be a n-person non-cooperative game. Consider player $i$'s position in this game. He is informed about the pure strategy sets of all the players. He is also aware of the von Neumann-Morgenstern utilities of all players associated with every possible n-tuple of pure strategies. The only other information he has about the other players is that they are rational players. The game is to be played just once. Given all these facts, which strategy should he play in order to maximize his utility? In this situation, if a logical analysis of the problem requires player $i$ to play a particular strategy or a strategy from a particular set of strategies, such a course of action can be called a solution for player $i$. On the other hand, a logical analysis of the situation under the given set of information may not lead to any particular conclusion, in which case we can say that for the given game, there is no solution for player $i$. In the latter case, assuming that not playing the game is not one of the options that player $i$ has, player $i$ is still faced with the question of having to pick a strategy. We will assume that in this case player $i$ recourses to a "standard of behavior" (as distinct from a solution) to pick a strategy from the set of all his strategies. Which standard of behavior player $i$ should opt for is then clearly a meta-game theoretical question and beyond the scope of game theory.

We will now attempt to define a solution for a non-cooperative game (if one exists). Consider again player $i$'s situation in a game. If he had prior information about the strategies that his opponents would employ, his problem of selecting a strategy would simplify to finding the strategy which would maximize his utility subject to the restriction that each of his opponents play a fixed strategy which is known to player $i$. However, player $i$ has no such prior information. The only clue he has about the actions of the other players is the fact that they are rational players. What does the assumption of rationality imply about players' behavior?

One implication is that if for some player $k$, his pure strategy $\pi_{k\alpha}$ is strongly dominated by another pure strategy $\pi_{k\beta}$, then player $k$ has never any incentive to play a mixed strategy that uses the pure strategy $\pi_{k\alpha}$. This is because, no matter what strategies the other players play, player $k$ can do better by playing instead the mixed strategy obtained by substituting $\pi_{k\beta}$ in place of $\pi_{k\alpha}$. Thus a given game can be reduced by the elimination of all strongly dominated pure strategies of all the players. The reduced game is ain examined for strongly dominated pure strategies and the process continued ntil no player has a strongly dominated pure strategy.

What else can we deduce from the assumption of rationality? We examine this irst for 2-person games. If player $i$ plays a mixed strategy $s^{*i}$, then the est reply for the other player, $j$, is to play any strategy from the set

$$M_j(s^{*i}) = \{s^{*j} : u_j(s^{*j},s^{*i}) = \max_{s^j} u_j(s^j,s^{*i})\} . \tag{3.1}$$

imilarly, if player $j$ plays a mixed strategy $s^{*j}$, the best reply for player is to play a strategy from the set $M_i(s^{*j})$ defined as in (3.1). Suppose, the basis of the assumption of rationality, we can rationalize a unique trategy $s^{*i}$ for player $i$. I.e., we suppose that, since player $i$ is a ational player, he is expected to play a particular strategy $s^{*i}$ (and no ther). Then, since player $j$ is also a rational player, we can rationalize he set of strategies $M_j(s^{*i})$ for player $j$. I.e., player $j$ can be expected play any strategy from the set $M_j(s^{*i})$. Then, if our original assumption of ationalizing $s^{*i}$ for player $i$ is to be valid, we must have

$$\{s^{*i}\} = M_i(s^j) \ \forall \ s^j \in M_j(s^{*i}) .$$

n general, we may be able to rationalize a (unique) set of strategies for each layer. We make the following formal definition for a 2-person game. A non-mpty set of strategies $X^i$ can be <u>rationalized</u> for player $i$ if and only if t is the unique set satisfying the following two conditions:

$$\exists \ X^j \text{ such that } X^j = M_j(s^i) \ \forall \ s^i \in X^i \tag{3.2}$$

$$X^i = M_i(s^j) \ \forall \ s^j \in X^j . \tag{3.3}$$

The following proposition is an obvious consequence of the above definition.

<u>roposition 3.1</u>. If $X^i$ can be rationalized for player $i$, then $X^j$ given by 3.2) can be rationalized for player $j$.

<u>roof</u>: Since conditions (3.2) and (3.3) are valid, we only need to show that $^j$ is a unique set satisfying these conditions. This follows from the fact hat $X^i$ is a unique set satisfying these conditions.

Q.E.D.

The concept of rationalizing a set of strategies for each player in a -person game can easily be generalized to a n-person game. Let

$$M_i(s^1,\ldots,s^{i-1},s^{i+1},\ldots,s^n) = \{t^i : u_i(\bar{s};t^i) = \max_{\text{all } r^i\text{'s}} [u_i(\bar{s};r^i)]\}$$

$$\text{where } \bar{s} = (s^1,\ldots,s^n) .$$

Let $\Gamma$ be an n-person game. Let $X = (X^1,\ldots,X^n)$ be an n-tuple of nonempty sets of strategies. We say $X$ can be rationalized for $\Gamma$ (or $X^i$ can be rationalized for player $i$, $i = 1,\ldots,n$) if $X$ is the unique n-tuple satisfyin for all $i \in N$

$$X^i = M_i(s^1,\ldots,s^{i-1},s^{i+1},\ldots,s^n) \;\forall\; (s^1,\ldots,s^{i-1},s^{i+1},\ldots,s^n) \in X^1 \times \ldots \times X^{i-1} \times X^{i+1} \times \ldots \times X^n .$$

Thus we see that the concept of rationalizing an n-tuple of sets of strategies for a game is a minimal condition that every solution of a non-cooperative game should satisfy, i.e., it is a "necessary" condition. We will now attempt to show that it is, in a sense, a "sufficient" condition as well.

Consider a 2-person game such that we can rationalize $X^i$ for player $i$ and $X^j$ for player $j$. Player $i$'s situation can be summarized as in Table 1. Hence player $i$ has a reasonable justification for playing a strategy from th set $X^i$. Also if player $j$ anticipates this action of player $i$, his subsequent action merely reinforces player $i$'s choice of picking a strategy from $X^i$. A similar argument can be made for player $i$ if the game has $n$ players.

| If player $i$ picks a strategy from the set | and | Assuming that player $j$ picks a strategy from the set | then | The utility payoff to player $i$ is |
|---|---|---|---|---|
| $X^i$ | | $X^j$ | | the best that playe $i$ can hope for |
| | | $(X^j)^c$ | | indeterminate |
| $(X^i)^c$ | | $X^j$ | | worse off than if player $i$ had played a strategy from $X^i$ |
| | | $(X^j)^c$ | | indeterminate |

Table 1

We have stated two implications of rationality. We can consider these as loms that a solution of a non-cooperative game should always satisfy (if one ists). For example,

lom 0: A non-cooperative game may or may not have a solution.

lom 1: If a non-cooperative game has a solution and $\bar{s}$ is an n-tuple of rategies in the solution, then $\bar{s}$ does not use any strongly dominated rategy.

lom 2: If a non-cooperative game has a solution, then it should be rationalible for the game.

It is clear from the definitions that a rationalizable set cannot contain a rategy that uses a strongly dominated strategy. Hence Axiom 2 implies Axiom 1. the next section, we examine Nash's various solutions and see how they relate our axioms.

IV. THE ROLE OF EQUILIBRIUM POINTS IN SOLUTIONS OF NON-COOPERATIVE GAMES.

e concept of a Nash equilibrium point is the basic ingredient of Nash's theory non-cooperative games. We will show that it also plays an important role in r theory.

oposition 4.1. Let $X$ be rationalizable for $\Gamma$. Then $\bar{s} \in X \Rightarrow \bar{s}$ is a Nash uilibrium point.

The proof follows from the definition of a rationalizable set for $\Gamma$. We w examine Nash's theory of non-cooperative games and see how they relate to r axioms.

eorem 4.2: Let $\Gamma$ be a strongly Nash solvable game. Then the strong Nash lution $S$ is rationalizable for $\Gamma$.

oof: Let $X^i = \{r^i : (\bar{s};r^i) \in S$ for some $\bar{s}\}$. Clearly

$$X^i \subset M_i(s^1,\dots,s^{i-1},s^{i+1},\dots,s^n) \;\forall\; (s^1,\dots,s^{i-1},s^{i+1},\dots,s^n)$$

$$\in X^1 \times \dots \times X^{i-1} \times X^{i+1} \times \dots \times X^n .$$

nce $\Gamma$ is strongly Nash solvable,

$$\bar{s} \in S\,,\; u_i(\bar{s};r^i) = u_i(\bar{s}) \Rightarrow (\bar{s};r^i) \in S\,.$$

we have

$$X^i \supset M_i(s^1,\dots,s^{i-1},s^{i+1},\dots,s^n) \;\forall\; (s^1,\dots,s^{i-1},s^{i+1},\dots,s^n)$$

$$\in X^1 \times \dots \times X^{i-1} \times X^{i+1} \times \dots \times X^n .$$

Hence

$$X^1 = M_i(s^1,\ldots,s^{i-1},s^{i+1},\ldots,s^n) \;\forall\; (s^1,\ldots,s^{i-1},s^{i+1},\ldots,s^n)$$

$$\in X^1 \times \ldots \times X^{i-1} \times X^{i+1} \times \ldots \times X^n .$$

Hence $X = (X^1,\ldots,X^n)$ is rationalizable for $\Gamma$. But $X = S$. Hence $S$ is rationalizable for $\Gamma$.

Q.E.D.

Theorem 4.2 states that strong Nash solvability is a sufficient condition for the existence of a rationalizable set and that the rationalizable set coincides with the strong Nash solution. However, the surprising result is that strong Nash solvability is not a necessary condition for the existence of a rationalizable set. The following example illustrates this fact.

<u>Example 4.1</u>: Consider the 2-person game represented by the matrix given below

| | | 2: $\beta_1$ | 2: $\beta_2$ |
|---|---|---|---|
| 1: | $\alpha_1$ | (1,3) | (1,3) |
| | $\alpha_2$ | (0,0) | (2,2) |

The equilibrium points of this game are $(\alpha_1,\beta_1)$ and $(\alpha_2,\beta_2)$. These are not interchangeable, hence the game is not even Nash solvable. However, it can easily be shown that $\{(\alpha_2,\beta_2)\}$ is rationalizable for the game. □

Since the game in Example 4.1 is not Nash solvable, Nash solvability is not a necessary condition for the existence of the rationalizable set. Moreover, Nash solvability is not a sufficient condition for the existence of a rationalizable set. This is shown in the next example.

<u>Example 4.2</u>: Consider the 2-person game represented by the matrix given below

| | | 2: $\beta_1$ | 2: $\beta_2$ |
|---|---|---|---|
| 1: | $\alpha_1$ | (5,-3) | (-4,5) |
| | $\alpha_2$ | (-5,5) | (3,-4) |

This game has a unique equilibrium point $(\frac{9}{16}\alpha_1 + \frac{7}{16}\alpha_2, \frac{7}{17}\beta_1 + \frac{10}{17}\beta_2)$. Thus the game is Nash solvable. The Nash value of the game to player 1 is -5/17 and to player 2 is 1/2. It can easily be shown that the rationalizable set does not xist for this game. Hence from our point of view, the game has no solution. o see why Nash's solution is not really a solution of this game, consider player 's position. If he plays his equilibrium strategy, the maximum he can get is is Nash value, 1/2, provided player 1 also plays his equilibrium strategy. Howver, player 2 can guarantee his Nash value irrespective of player 1's actions by imply playing the maximin strategy $(\frac{1}{2}\beta_1 + \frac{1}{2}\beta_2)$. Moreover, if player 2 plays is equilibrium strategy and player 1 plays his maximin strategy $(\frac{8}{17}\alpha_1 + \frac{9}{17}\alpha_2)$ to guarantee his Nash value, -5/17), player 2 actually gets 107/289 which is ess than his Nash value!

On the subject of rational behavior, von Neumann and Morgenstern [9] write:

> "... the rules of rational behavior must provide definitely for the possibility of irrational conduct on the part of others... . If that should turn out to be advantageous for them - and quite particularly, disadvantageous to the conformists then the above "solution" would seem very questionable".

lence it is not clear why player 2 should play his equilibrium strategy. □

Next, we study the implications of our axioms when applied to the special nd well known case of 2-person zero-sum games. We say a 2-person zero-sum game as a saddle point if it has an equilibrium point in pure strategies. I.e. if $\pi_{i\alpha}, \pi_{j\beta}$ such that $(\pi_{i\alpha}, \pi_{j\beta})$ is an equilibrium point.

roposition 4.3. Let $\Gamma$ be a 2-person zero-sum game. The game has a rational-zable set only if $\Gamma$ has a saddle point.

roof: If $\Gamma$ has a saddle point such that it is a strong Nash solution, then y Theorem 4.2 it is rationalizable for $\Gamma$. If $\Gamma$ has no saddle point, then here exists a unique Nash equilibrium in mixed strategies. If player $i$ plays is equilibrium strategy, then player $j$ can play any pure strategy used in his quilibrium strategy and still get his Nash value of the game and vice-versa. lence $\nexists$ no rationalizable set for the game.

Q.E.D.

Thus, as per our theory, a 2-person zero-sum game with no saddle point has no solution. This is in sharp contrast with the universally accepted theory of von Neumann and Morgenstern [9] that the equilibrium point always constitutes a solution of a 2-person zero-sum game. Although we agree that there are many other reasons why a player may want to play the equilibrium strategy†, we feel that it is not necessarily a consequence of the assumption of rationality of the players.

Since the rationalizable set does not always exist, we cannot have a general existence result. However, this should not be interpreted negatively. I.e. a lack of a general existence result is not a "defect" in our theory. It is merely

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†Some of these reasons are discussed in Section V of this paper.

an outcome of the "lack of information" that a player has in playing certain non-cooperative games. I.e. some games, those for which a rationalizable set does not exist, do not give sufficient insight into the behavior of players assuming only rationality. We do not believe that the conditions imposed by Axiom 2 are too strong and must therefore be modified to admit existence for all games. We feel that Axiom 2 is a minimal condition that every solution should satisfy. For a game that has no solution (in our sense), a player can recourse to a "standard of behavior". These are discussed in the next section.

V. SOME STANDARDS OF BEHAVIOR. Let $\Gamma$ be a game that has no rationalizable set. Consider the position of a player, i. He has to pick a strategy to maximize his utility. His job is complicated by the fact that since the rationalizable set does not exist, he has no inkling of the strategies that the other players are going to pick. Some of the possible actions that he can take are as follows.

Undominated Strategies.

The fact that the game has no rationalizable set does not exclude the fact that some player(s) may have strongly dominated pure strategies. If this is the case, it is safe to assume that a player will never use a strongly dominated pure strategy in any mixed strategy and thus the game can be reduced by the elimination of all strongly dominated pure strategies. The reduced game is again examined for strongly dominated pure strategies and the process continued until no player has a strongly dominated pure strategy. At the end of this reduction process, since the game has no rationalizable set, there will be at least 2 players each of whom will have at least 2 pure strategies.

Let $\Gamma$ be a game with no rationalizable set and no strongly dominated pure strategy. Suppose some player, j, has a weakly dominated pure strategy. Since player j can do as well (if not better) by substituting the weakly dominated pure strategy by the dominating pure strategy in any mixed strategy that uses such a weakly dominated strategy, it is conceivable that he will never use his weakly dominated pure strategy in any mixed strategy. Thus the game can be reduced by the elimination of all weakly dominated strategies. By the same reasoning, the reduced game is again examined for weakly dominated strategies and the process continued until no player has a weakly dominated strategy.

Maximin Strategies.

In a finite game, maximin strategies always exist for all players. Let $\Gamma$ be a game for which no rationalizable set exists. Also suppose that no player has a dominated pure strategy. For such games, since a player has no idea of the strategies that the other players will play, he may decide to protect himself as much as possible by playing the maximin strategy. Thus by playing a maximin strategy, a player, i, is assured of getting at least his maximin value $v_i^m$ irrespective of the actions of the other players.

For 2-person zero-sum games, a player's maximin strategy is also his minimax trategy since

$$\max_{s^i} \min_{s^j} [u_i(s^i,s^j)] = \max_{s^i} \min_{s^j} [-u_j(s^i,s^j)]$$

$$= \max_{s^i} \{-\max_{s^j} [u_j(s^i,s^j)]\}$$

$$= -\{\min_{s^i} \max_{s^j} [u_j(s^i,s^j]\} .$$

lso since for all 2-person zero-sum games,

$$v_i^m = -v_j^m$$

player's maximin strategy is also his equilibrium strategy. Thus, in a -person zero-sum game, there is a strong motivation for a player to play his aximin (which is also his minimax and equilibrium) strategy. However, as menioned before, we are not willing to subscribe to the theory that this constintes a solution of the game.

In general, for 2-person non-zero-sum games, maximin strategies are distinct rom equilibrium strategies and often the maximin value of a player is equal to he Nash value (when it exists). In such cases we feel that it is better in ome respects for a player to play his maximin strategy instead of his equilibium strategy.

inimax Strategies in 2-Person Games.

For 2-person non-zero-sum games, minimax strategies are usually distinct rom maximin strategies. However they often coincide with equilibrium strateies. Since in a non-zero-sum game, the utility of an outcome for a player has o relation to the utility of the same outcome to his opponent, we cannot see ny motivation for a rational player to play his minimax strategy (on its' erits alone).

quilibrium Strategies.

Since equilibrium points always exist, every player $i$ has a nonempty set $_i$ of equilibrium strategies. The concept of an equilibrium strategy alone is ot strong enough to qualify even as a standard of behavior. E.g., for games hat are not Nash solvable, it makes no sense for a player to play an equilibium strategy because the resulting outcome may not be an equilibrium point. or games that are Nash solvable (but not strongly Nash solvable) equilibrium trategies may qualify as a standard of behavior.

We end this section by discussing a 2-person non-zero-sum game in detail.

| | | | 2 | | | |
|---|---|---|---|---|---|---|
| | | | equilibrium | equilibrium | equilibrium and minimax | maximin |
| | | | $\beta_1$ | $\beta_2$ | $3/8\ \beta_1 + 5/8\ \beta_2$ | $5/8\ \beta_1 + 3/8\ \beta_2$ |
| 1 | equilibrium | $\alpha_1$ | (1,2) | (-1,-4) | (-1/4,-7/4) | (1/4,-1/4) |
| | equilibrium | $\alpha_2$ | (-4,-1) | (2,1) | (-1/4,1/4) | (-7/4,-1/4) |
| | equilibrium and minimax | $1/4\ \alpha_1 + 3/4\ \alpha_2$ | (-11/4,-1/4) | (5/4,-1/4) | (-1/4,-1/4) | (-5/4,-1/4) |
| | maximin | $3/4\ \alpha_1 + 1/4\ \alpha_2$ | (-1/4,5/4) | (-1/4,-11/4) | (-1/4,-5/4) | (-1/4,-1/4) |

Table 2 . A Summary of Some of the Options Available to Player 1 & 2 and Their Consequences.

Example 5.1. Consider the 2-person game represented by the matrix given below.

| | | 2: $\beta_1$ | 2: $\beta_2$ |
|---|---|---|---|
| 1 | $\alpha_1$ | (1,2) | (-1,-4) |
| 1 | $\alpha_2$ | (-4,-1) | (2,1) |

This game has no dominated strategies and also no rationalizable set. There are 3 equilibrium points, $(\alpha_1,\beta_1)$, $(\alpha_2,\beta_2)$ and $(\frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2, \frac{3}{8}\beta_1 + \frac{5}{8}\beta_2)$. Since these are not interchangeable, the game is not Nash solvable. The minimax strategy for player 1 is $(\frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2)$ and for player 2 is $(\frac{3}{8}\beta_1 + \frac{5}{8}\beta_2)$. The maximin strategy for player 1 is $(\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2)$ and for player 2 is $(\frac{5}{8}\beta_1 + \frac{3}{8}\beta_2)$. The maximin value for player 1 is -1/4 and for player 2 is -1/4. A summary of the various options open to player 1 and 2 and their consequences is shown in Table 2. If player 1 plays his equilibrium strategy $(\frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2)$ and player 2 plays his maximin strategy (to guarantee himself a payoff of -1/4), then player 1 gets only -1 1/4 whereas he can guarantee himself a payoff of -1/4 by playing his maximin strategy. Player 2 is in an identical situation. We let the reader judge for himself which strategy he would choose if he had to play the above game just once in the position of player 1 (or player 2) against a rational (but otherwise unknown) opponent.

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