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# Ambiguity Aversion and a Decision-Theoretic Framework Using Belief Functions

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***Abstract*—This paper introduces a new approach to constructing normative models that exhibit the same ambiguity aversion as human decision makers. The models are constructed using a decision-theoretic framework based on the theory of belief functions interpreted as generalized probability. The level of ambiguity aversion is determined by a subjective parameter in the unit interval so that users have the possibility to fix its strength in the model. We show using three examples that the decisions, which are determined by the optimization of a total subjective reward (corresponding to a reduced expected reward), are consistent with the experimental results observed by Ellsberg and other authors.**

***Index Terms*—belief functions, credal set, probability transform, decision-making, vagueness**

## I. Introduction

In this note, we propose a new way of modeling ambiguity aversion behavior using a theory of belief functions. Ellsberg [5] shows that the ambiguity aversion behavior violates the postulates of Savage's expected utility theory. Although we do not claim that the ambiguity aversion behavior is *rational*, we do consider it normal, i.e., non-paradoxical. Our subjective decision making behavior, and also that of many of our colleagues, is consistent with this phenomenon.

Savage's expected utility theory [13] is based on the idea that the behavior of a decision maker can be modeled with the help of a subjective probability mass function. In this paper, the models proposed are based on the theory of belief functions that can be interpreted as *generalized probability*, i.e., a set of probability mass functions (PMFs) called a credal set [8]. We adopt a decision-theoretic framework where a belief function is first transformed into a PMF, and then, to make a decision, we subscribe to Savage's expected utility theory. In our modification, however, we add one additional step. Before computing the expected reward, we reduce the probabilities to account for ambiguity aversion. Otherwise, our approach is similar to Smets' decision-making framework [16], which is based on the Dempster-Shafer theory of belief functions [4], [14]. Here, we are concerned with a theory of belief functions interpreted as generalized probability [8].

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We believe that the ambiguity aversion phenomenon is closely connected to the fact that classical probability theory has difficulties with representing ignorance, or vagueness [14]. This insufficiency explains why some decision theorists consider human decision-making behavior paradoxical. Consider the following simple example.

*a)* **6-Color Example:** *Consider an urn possibly containing balls of six colors:* red ($r$), blue ($b$), green ($g$), orange ($o$), white ($w$), *and* yellow ($y$). *You have neither any information about how many balls of each particular color are in the urn, nor about the total number of balls in it. You only know that there is at least one ball in the urn. You win \$100 if a randomly drawn ball is red. How much are you willing to pay to play this gamble?*

The fact that people are not willing to pay in the described situation as much as in the case where they know that the urn contains the same amount of balls of each color is called Ellsberg's paradox [5]–[7], or *ambiguity aversion*. Most people do not like making decisions under ignorance. They distinguish between two situations: knowing that the urn contains the same number of balls of each color (risk), versus the situation described in 6-Color Example (ambiguity). Perhaps this fact was also among the main reasons why several alternative uncertainty theories were developed in the last decades of the last century.

One of the theories designed to model vagueness is a theory of belief functions where the values of belief functions are interpreted as lower bounds on some unknown probabilities. The main goal of this paper is to describe a decision-making framework using the theory of belief functions yielding the outcomes corresponding to observed human behavior, not only in situations described by the 6-Color Example but also in the following example.

*b)* **One Red Ball Example:** *Consider an urn possibly containing balls of six colors:* red ($r$), blue ($b$), green ($g$), orange ($o$), white ($w$), *and* yellow ($y$). *The only information you have is $n$, the total number of balls in the urn, and also that there is exactly one red ball among them. A ball is randomly drawn from the urn, and you win \$100 if the color of the ball drawn from the urn matches the color that you chose before the draw. What color do you choose?*

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We are not proposing a psychological theory explaining human behavior. We are proposing a decision-theoretic framework that is descriptive of human behavior, especially in situations where it does not conform to Savage's expected utility theory. The outcomes of our framework are descriptive of the observations we made in ad-hoc discussions with students and colleagues. On a number of occasions, we presented the *One Red Ball Example* to an audience (usually students or researchers interested in AI) and asked them to describe how they would behave. We have the following observations:

- As one can expect, for $n \leq 6$, everyone chose red.
- Naturally, the situation changes with increasing $n$. For $n$ large enough no one chose red.
- Interesting situations are for $n = 7, \ldots, 11$. We do not have exact experimental data, but in ad hoc experiments, all chose the red color for $n = 7$. With increasing $n$, they usually quit betting on the red color for $n = 8$, or 9. But there were a few individuals who kept betting on the red ball even for $n = 10$ and 11!

This behavior was observed with students and researchers familiar with Savage's expected utility decision theory. In the general population, we can expect to encounter individuals who would keep choosing the red color even for higher values of $n$.

As we mentioned earlier, we believe that this fact is closely connected with the ambiguity aversion phenomenon. Nevertheless, for some individuals, it can also have a pragmatic basis. Not knowing the process of how the balls were inserted into the urn, they may be suspicious that it was done deliberately so that no one (or only a small minority) can win the reward. Whether or not there is a normative model that explains ambiguity aversion, it is a psychological phenomenon. Decision under vague (or incomplete) information is subjective, and therefore we should expect that a normative model would also have a subjective element.

The remainder of the paper is organized as follows. In Section II, we make a brief introduction to belief function theory and its basic interpretation. In Section III we introduce three different approaches to transforming a belief function to an equivalent PMF. In Section IV we propose a new way of computing a subjective expected reward under the ambiguity aversion, and Section V illustrates this approach on two examples presented above, and on the example from the Ellsberg's paper [5]. In fact, it is this Section V where the reader learns how to proceed to get models with the required properties. The last section concludes with a summary and some unsolved issues.

## II. Belief Functions

It is beyond the scope of this paper to introduce the full theory of belief functions. For this, the interested reader is referred to [14]. Fortunately, in this paper, it is sufficient to introduce a couple of basic notions from this theory. Similar to probability theory, where a probability measure is a set function defined on some $\sigma$-algebra, belief functions are represented by functions defined on the set of all nonempty subsets of a state space $\Omega$. We assume $\Omega$ is finite. Let $2^{\Omega}$ denote the set of all *nonempty* subsets of $\Omega$.

We start with a definition of a *basic probability assignment* (bpa). It is a function $m : 2^{\Omega} \to [0, 1]$, such that

$$\sum_{\mathbf{a} \in 2^{\Omega}} m(\mathbf{a}) = 1.$$

$\mathbf{a} \in 2^{\Omega}$ is said to be a *focal element* of bpa $m$ if $m(\mathbf{a}) > 0$. Based on the set of focal elements we describe two classes of special bpa's:

- $m$ is said to be *vacuous* if $m(\Omega) = 1$, i.e., it has only one focal element, $\Omega$. A vacuous bpa is denoted by $m_{\iota}$.
- $m$ is said to be *Bayesian*, if all its focal elements are singletons, i.e., for Bayesian bpa $m$, $m(\mathbf{a}) > 0$ implies $|\mathbf{a}| = 1$.

Vacuous bpa $m_{\iota}$ represents total ignorance, whereas Bayesian bpa's represents exactly the same type of knowledge as PMFs. As all focal elements of a Bayesian bpa $m$ are singletons, we can define the PMF $P_m$ for $\Omega$ corresponding to $m$ such that

$$P_m(x) = m(\{x\}) \tag{1}$$

for all $x \in \Omega$.

In the generalized probability theory of belief functions, which is what we are concerned with in this paper, the fact that $\mathbf{a} \subseteq \Omega$, for which $|\mathbf{a}| > 1$, is a focal element for bpa $m$, expresses our ignorance regarding how the probability mass $m(\mathbf{a})$ is distributed among the elements of set $\mathbf{a}$. For example, suppose $\Omega = \{x_1, x_2\}$, and bpa $m$ is defined as follows: $m(\{x_1\}) = 0.2$, $m(\{x_2\}) = 0.3$, $m(\{x_1, x_2\}) = 0.5$. Bpa $m$ represents the knowledge that the probability of $x_1$ is at least 0.2 and at most 0.7, and the probability of $x_2$ is at least 0.3 and at most 0.8. We know nothing more, nothing less.

The same knowledge that is expressed by a bpa $m$ can also be expressed by a belief function, and by a plausibility function. Namely, each bpa $m$ can be uniquely characterized by the *belief function* $Bel_m$ corresponding to $m$, which is for all $\mathbf{a} \in 2^{\Omega}$ defined as follows:

$$Bel_m(\mathbf{a}) = \sum_{\mathbf{b} \in 2^{\Omega}:\, \mathbf{b} \subseteq \mathbf{a}} m(\mathbf{b}). \tag{2}$$

Alternatively, each bpa $m$ can also be uniquely characterized by the *plausibility function* $Pl_m$ corresponding to $m$, which is for all $\mathbf{a} \in 2^{\Omega}$ defined as follows:

$$Pl_m(\mathbf{a}) = \sum_{\mathbf{b} \in 2^{\Omega}:\, \mathbf{b} \cap \mathbf{a} \neq \emptyset} m(\mathbf{a}). \tag{3}$$

Notice that from (2) and (3), it is obvious that for all $\mathbf{a} \in 2^{\Omega}$,

$$Bel(\mathbf{a}) \leq Pl(\mathbf{a}).$$

If $Bel(\mathbf{a}) = Pl(\mathbf{a})$ then we are sure that the probability of $\mathbf{a}$ equals this value. Otherwise, a larger difference $Pl(\mathbf{a}) - Bel(\mathbf{a})$ means there is more ambiguity about the value of the probability of $\mathbf{a}$.

## III. Probability Transforms of Belief Functions

Savage's decision-making theory is based on the computation of expected utility. To compute it we need a PMF. Therefore, given a bpa $m$, we need a way to find a PMF that represents $m$. Equation (1) describes a unique relation between PMFs and Bayesian bpa's. A non-Bayesian bpa $m$ corresponds to a convex set of PMFs $P$ on $\Omega$ as follows ($\mathcal{P}$ denote the set of all PMFs on $\Omega$):

$$\mathcal{P}(m) = \left\{ P \in \mathcal{P} : \sum_{x \in \mathsf{a}} P(x) \geq Bel_m(\mathsf{a}) \text{ for } \forall \mathsf{a} \in 2^{\Omega} \right\}.$$

$\mathcal{P}(m)$ is called the *credal set* of bpa $m$. Notice that $P_m$ defined by Equation (1) for a Bayesian bpa $m$ is such that $\mathcal{P}(m) = \{P_m\}$. In addition to this, there are several methods for selecting one PMF that represents bpa $m$. By *probability transform* we mean a mapping that assigns a PMF to each bpa. In this paper, we consider three such probability transforms. For their properties and for other probability transforms not discussed in this paper, the reader is referred to [3].

One straightforward approach is to select the maximum entropy representative of the respective credal set, i.e.,

$$Me\_P_m = \arg \max_{P \in \mathcal{P}(m)} \{H(P)\},$$

where $H(P)$ denotes the Shannon entropy of PMF $P$.

Another representative, called the *pignistic transform*, was advocated by Smets [15], [17]. It is defined for all $x \in \Omega$ as follows:

$$Bet\_P_m(x) = \sum_{\mathbf{a} \in 2^{\Omega} : x \in \mathbf{a}} \frac{m(\mathbf{a})}{|\mathbf{a}|}.$$

Notice that this transform redistributes the probability mass assigned to a non-singleton set $\mathbf{a}$ equally to all singletons contained in $\mathbf{a}$.

The last transform considered in this paper is the *plausibility transform* suggested by Voorbraak [18], which is defined for all $x \in \Omega$ as follows:

$$Pl\_P_m(x) = \frac{Pl_m(\{x\})}{\sum_{y \in \Omega} Pl_m(\{y\})}.$$

As advocated by Cobb and Shenoy [2], this is the only transform consistent with the Dempster-Shafer (DS) theory of evidence, because it is the only transform that commutes with the Dempster combination rule. As we are concerned with the generalized probability theory of belief function and not DS theory, we do not go into details and refer the interested reader to papers [2], [3].

A summary of the basic properties of the three probability transforms, which are important from the point of view of this paper, are as follows:

- For vacuous bpa $m_\iota$, $Me\_P_{m_\iota}(x) = Bet\_P_{m_\iota}(x) = Pl\_P_{m_\iota}(x) = \frac{1}{|\Omega|}$ for all $x \in \Omega$.
- For Bayesian bpa $m$, $Me\_P_m = Bet\_P_m = Pl\_P_m = P_m$, where $P_m$ is from (1).
- Both $Me\_P_m$ and $Bet\_P_m$ belong to $\mathcal{P}(m)$. It does not necessarily hold for $Pl\_P_m$.

## IV. Rewards and Subjective Reduced Weights

This section describes how a belief function model can be used to get an optimal decision that imitates the human way of decision-making.

Suppose our uncertainty is described by a general bpa $m$ on $\Omega$. First, we follow a standard approach and transform bpa $m$ to PMF[1] $P_m$ defined on $\Omega$. For this, any transform described in the preceding section can be used. Nevertheless, in the next section, where we present some examples, we will see that the preferable transform is the pignistic transform. The pignistic transform is used in Smets' transferable belief model for decision making [17]. Our decision-making framework is similar to Smets', but we are using the generalized probability semantics of belief functions instead of Smets' transferable belief model semantics.

Usually, decision makers choose an alternative that is optimal with respect to PMF $P_m$. The novelty of our approach lies in the following simple idea. Inspired by Hurwicz's *optimism-pessimism* approach [9], [10], we introduce a subjective *coefficient of ambiguity aversion* $\alpha \in [0,1]$. The higher the value of this coefficient, the higher the ambiguity aversion of a decision maker. This coefficient is then used to reduce the expected reward. We do not propose to compute an expected value. Instead, we propose to compute a weighted sum of the rewards. For this computation, we do not use probabilities, but some weights, which we call $r$-weights (for *reduced* weights), which do not sum to one. These weights are defined as follows:

$$r_{m,\alpha}(x) = (1-\alpha)P_m(x) + \alpha Bel_m(\{x\}) \tag{4}$$

for all $x \in \Omega$. Notice that each $r$-weight $r_{m,\alpha}(x)$ can be regarded as a reduced version of probability $P_m(x)$. The amount of reduction depends on the ambiguity aversion coefficient $\alpha$, and the amount of ignorance associated with the state $x$. If we are certain about the probability of state $x$, it means that $P_m(x) = Bel_m(\{x\})$, and the corresponding probability is not reduced: $r_{m,\alpha}(x) = P_m(x)$. On the other hand, the maximum reduction is achieved for the states connected with maximal ambiguity, i.e., for the states for which $Bel_m(\{x\}) = 0$.

Some trivial properties of $r$-weights are as follows:

1) $\sum_{x \in \Omega} r_{m,\alpha}(x) \leq 1$; and
2) $m$ is Bayesian if and only if $m(\{x\}) = P_m(x) = r_{m,\alpha}(x)$ for all $x \in \Omega$, and $\alpha \in [0,1]$.

These $r$-weights are then used to compute *total subjective reward*, which is computed similarly to expected value, but the probabilities are substituted by the respective $r$-weights.

$$R_{m,\alpha} = \sum_{x \in \Omega} r_{m,\alpha}(x) g(x),$$

where $g(x)$ denote the reward (gain) one expects in case $x \in \Omega$ occurs. Thus, $R_{m,\alpha}$ does not express a mathematical expected reward, but a subjectively reduced expectation of a decision maker, whose subjectivity, i.e., level of ambiguity aversion,

[1] Notice that we used the same symbol as that in (1). It does not lead to any confusion because if $m$ is Bayesian then all the introduced probability transforms yield the PMF defined in (1).

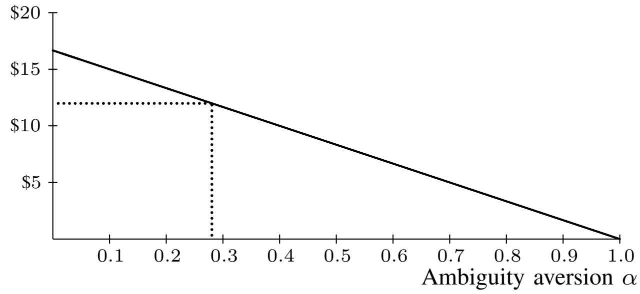


Fig. 1. Maximal bets in 6-Color Example in dependence on the coefficient of ambiguity aversion

is described by $\alpha$. In a way, it corresponds to what is called subjective expected utility (SEU) by other authors [1], [11].

## V. Three Examples

*a)* **6-Color Example:** For this example, $\Omega = \{r, b, g, o, y, w\}$, and the knowledge about $\Omega$ is described by the vacuous bpa $m_\iota$ for $\Omega$.

In this case $Me\_P_{m_\iota} = Bet\_P_{m_\iota} = Pl\_P_{m_\iota} = P_{m_\iota}(x) = \frac{1}{6}$ and $r_{m_\iota,\alpha}(x) = \frac{1-\alpha}{6}$ for all colors $x \in \Omega$. Let $g(x)$ denote the gain received in case when color $x$ is drawn, i.e., $g(r) = 100$, and for $x \neq r$, $g(x) = 0$. The total subjective reward is as follows:

$$R_{m_\iota,\alpha} = \sum_{x\in\Omega} r_{m_\iota,\alpha}(x) g(x) = \sum_{x\in\Omega} \frac{1-\alpha}{6} g(x) = \frac{100\cdot(1-\alpha)}{6}.$$

This can be interpreted as follows. A person should be willing to pay (for playing the game) a maximum amount of \$ $\frac{100\cdot(1-\alpha)}{6}$. Thus, a person with $\alpha = 0.28$ is willing to pay a maximum of \$12 (see the graph in Figure 1).

*b)* **One Red Ball Example:** For this example, again $\Omega = \{r, b, g, o, y, w\}$, and the uncertainty is described by the bpa $m_\varrho$ as follows:

$$m_\varrho(\mathbf{a}) = \begin{cases} \frac{1}{n}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{n-1}{n}, & \text{if } \mathbf{a} = \{b, g, o, y, w\}; \\ 0, & \text{otherwise.} \end{cases}$$

First, let us apply the plausibility transform. The respective plausibility function for all singletons is as follows:

$$Pl_{m_\varrho}(\{x\}) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ \frac{n-1}{n}, & \text{for } x \in \{b, g, o, y, w\}, \end{cases}$$

which means that

$$\sum_{x\in\Omega} Pl_{m_\varrho}(\{x\}) = \frac{5n-4}{n},$$

and, therefore,

$$Pl\_P_{m_\varrho}(x) = \begin{cases} \frac{1}{5n-4}, & \text{if } x = r; \\ \frac{n-1}{5n-4}, & \text{for } x \in \{b, g, o, y, w\}. \end{cases}$$

The $r$-weights are as follows:

$$r_{m_\varrho,\alpha}(x) = \begin{cases} \frac{1}{5n-4}, & \text{if } x = r; \\ \frac{(1-\alpha)(n-1)}{5n-4}, & \text{for } x \in \{b, g, o, y, w\}, \end{cases}$$

because $Bel_m(x) = 0$ for all $x \in \{b, g, o, y, w\}$.

For simplicity's sake, let us consider betting on red and white colors. For these two colors, the respective gain functions are denoted $g^r(x)$, and $g^w(x)$, respectively (i.e., $g^r(r) = 100$, and for $x \neq r$, $g^r(x) = 0$, and, analogously, $g^w(w) = 100$, and for $x \neq w$, $g^w(x) = 0$). This yields the total subjective rewards when betting on red as follows:

$$R_{m_\varrho,\alpha}(r) = \frac{1}{5n-4} g^r(r) + \sum_{x\in\Omega:x\neq r} \frac{(1-\alpha)(n-1)}{5n-4} g^r(x) = \frac{100}{5n-4},$$

and analogously, for betting on white

$$R_{m_\varrho,\alpha}(w) = \frac{1}{5n-4} g^w(r) + \sum_{x\in\Omega:x\neq r} \frac{(1-\alpha)(n-1)}{5n-4} g^w(x) = \frac{100(1-\alpha)(n-1)}{5n-4}.$$

This means that if $\alpha = 0$, i.e., no ambiguity aversion, we would never bet on the red ball if $n > 2$. For three balls in the urn ($n = 3$) we would bet on the red ball only with the coefficient of ambiguity $\alpha > \frac{1}{2}$, and for $n = 5$ we would bet on the red ball only with $\alpha > \frac{3}{4}$. In our opinion, this does not correspond to our informal experimental findings.

However, using the pignistic transform (or maximum entropy, in this example these two transforms coincide), we get:

$$Bet\_P_{m_\varrho}(\{x\}) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ \frac{n-1}{5n}, & \text{for } x \in \{b, g, o, y, w\}. \end{cases}$$

Since $Bel_{m_\varrho}(\{x\}) = 0$ for all $x \in \{b, g, o, y, w\}$, and $Bel_{m_\varrho}(\{r\}) = \frac{1}{n}$ we get the following reduced weights:

$$r_{m_\varrho,\alpha}(x) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ (1-\alpha)\cdot\frac{n-1}{5n}, & \text{for } x \in \{b, g, o, y, w\}. \end{cases}$$

Considering the gain functions $g^r(x)$, and $g^w(x)$, the total subjective rewards are as follows. When betting on red it equals

$$R_{m_\varrho,\alpha}(r) = \frac{1}{n} g^r(r) + \sum_{x\in\Omega:x\neq r} \frac{(1-\alpha)(n-1)}{5n} g^r(x) = \frac{100}{n},$$

and analogously, for betting on white

$$R_{m_\varrho,\alpha}(w) = \frac{1}{n} g^w(r) + \sum_{x\in\Omega: x\neq r} \frac{(1-\alpha)(n-1)}{5n} g^w(x)$$
$$= \frac{100(1-\alpha)(n-1)}{5n}.$$

Some of the values of these functions are tabulated in Table I. From this table we see that, for example, a person with $\alpha = 0.28$ should bet on red color for $n \leq 7$, because for these $R_{m_\varrho,\alpha}(r) > R_{m_\varrho,\alpha}(x)$ ($x \neq r$), and bet on any other color for $n \geq 8$, because for these $n$, $R_{m_\varrho,\alpha}(r) \leq R_{m_\varrho,\alpha}(x)$ ($x \neq r$). This means that for $n \leq 7$, it is subjectively more advantageous to bet on the red color.

*c)* **Ellsberg's Example:** Consider the original Ellsberg's example ( [5], pp. 653–654) with an urn containing 30 red balls and 60 black or yellow balls, the latter in unknown proportion. With this urn, Ellsberg considers two experiments. The first experiment (Ellsberg's Actions I and II) studies whether people prefer betting on red or black ball, in which case they get the reward ($100) if the ball of the respective color is drawn at random. In the second experiment (Ellsberg's Actions III and IV), a person has a possibility to bet on red and yellow, or, alternatively, on black and yellow. Again, the participant gets the reward ($100) in case that the randomly drawn ball is of one of the selected colors.

The uncertainty can be described with $\Omega = \{r, b, y\}$ and the bpa $m_\varepsilon$ as follows:

$$m_\varepsilon(\mathbf{a}) = \begin{cases} \frac{1}{3}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{2}{3}, & \text{if } \mathbf{a} = \{b, y\}; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that the pignistic transform (notice that it is again the same as the maximum entropy transform and different from the plausible transform) is a uniform PMF $Bet\_P_{m_\varepsilon}(x) = \frac{1}{3}$ for all $x \in \Omega$. The corresponding belief function is $Bel_{m_\varepsilon}(\{r\}) = \frac{1}{3}$, and $Bel_{m_\varepsilon}(\{b\}) = Bel_{m_\varepsilon}(\{y\}) = 0$. Therefore, we get the following $r$-weights

$$r_{m_\varepsilon,\alpha}(x) = \begin{cases} \frac{1}{3}, & \text{if } x = r; \\ \frac{(1-\alpha)}{3}, & \text{for } x \in \{b, y\}. \end{cases}$$

The gain functions $g^r(x)$, and $g^b(x)$ for betting on red and black balls, respectively, are as follows:

$$g^r(r) = 100, g^r(b) = g^r(y) = 0,$$
$$g^b(b) = 100, g^b(r) = g^b(y) = 0.$$

The total subjective reward for betting on red ball is as follows:

$$R_{m_\varepsilon,\alpha}(r) = \frac{1}{3} g^r(r) = \frac{100}{3},$$

and analogously, for betting on black ball is as follows:

$$R_{m_\varepsilon,\alpha}(b) = \frac{(1-\alpha)}{3} g^b(b) = \frac{100(1-\alpha)}{3}.$$

Thus, for positive $\alpha$, we get $R_{m_\varepsilon,\alpha}(r) > R_{m_\varepsilon,\alpha}(b)$, which is consistent with Ellsberg's observation that "very frequent pattern of response is that betting on red is preferred to betting on black."

Let us consider the second experiment, which involves betting on a couple of colors. In comparison with the first experiment, the situation changes only in the respective gain functions; denote them $g^{ry}(x)$ and $g^{by}(x)$ for betting on red and yellow, and for betting on black and yellow balls, respectively.

$$g^{ry}(r) = g^{ry}(y) = 100, g^{ry}(b) = 0,$$
$$g^{by}(b) = g^{by}(y) = 100, g^{by}(r) = 0.$$

Thus, the total subjective rewards are as follows:

$$R_{m_\varepsilon,\alpha}(ry) = \frac{1}{3} g^{ry}(r) + \frac{(1-\alpha)}{3} g^{ry}(y) = \frac{100(2-\alpha)}{3},$$

and similarly,

$$R_{m_\varepsilon,\alpha}(by) = \frac{(1-\alpha)}{3} g^{by}(b) + \frac{(1-\alpha)}{3} g^{by}(y)$$
$$= \frac{100(2-2\alpha)}{3}.$$

In this case we get $R_{m_\varepsilon,\alpha}(ry) \geq R_{m_\varepsilon,\alpha}(by)$, which is not consistent with Ellsberg's observation that "betting on black and yellow is preferred to betting on red and yellow balls." However, the decision maker has another way for evaluating the total subjective rewards $R_{m_\varepsilon,\alpha}(ry), R_{m_\varepsilon,\alpha}(by)$, which will be described in the rest of this section.

Compared to probabilities, the r-weights are not additive. So, thanks to the fact that we have beliefs for all subsets of $\Omega = \{r, b, y\}$, we can, using the idea expressed in formula (4), compute the r-weights (the reduced probabilities) not only for individual colors, but also for all nonempty subsets of $\Omega = \{r, b, y\}$. Notice that $Bel_{m_\varepsilon}(\{r\}) = \frac{1}{3}$, and $Bel_{m_\varepsilon}(\{b\}) = Bel_{m_\varepsilon}(\{y\}) = 0$, $Bel_{m_\varepsilon}(\{r, b\}) = Bel_{m_\varepsilon}(\{r, y\}) = \frac{1}{3}$, $Bel_{m_\varepsilon}(\{b, y\}) = \frac{2}{3}$, and $Bel_{m_\varepsilon}(\Omega) = 1$. Therefore,

$$r_{m_\varepsilon,\alpha}(\mathbf{a}) = \begin{cases} \frac{1}{3}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{(1-\alpha)}{3}, & \text{for } \mathbf{a} = \{b\}, \{y\}; \\ \frac{(2-\alpha)}{3}, & \text{for } \mathbf{a} = \{r, b\}, \{r, y\}; \\ \frac{2}{3}, & \text{if } \mathbf{a} = \{b, y\}. \end{cases}$$

Notice that the expected value

$$\sum_{x\in\Omega} Bet\_P_{m_\epsilon}(x) g^{ry}(x)$$
$$= Bet\_P_{m_\epsilon}(\{r, y\}) g^{ry}(r) + Bet\_P_{m_\epsilon}(b) g^{ry}(b),$$

because $g^{ry}(y) = g^{ry}(r)$. This suggests that we compute:

$$R_{m_\varepsilon,\alpha}(ry) = r_{m_\varepsilon,\alpha}(\{r, y\}) g^{ry}(r) + r_{m_\varepsilon,\alpha}(\{b\}) g^{ry}(b)$$
$$= \frac{(2-\alpha)}{3} 100,$$

TABLE I
ONE RED BALL EXAMPLE: TOTAL SUBJECTIVE REWARD AS A FUNCTION OF THE COEFFICIENT OF AMBIGUITY AVERSION $\alpha$, AND THE NUMBER OF BALLS $n$.

| | | $R_{m_\varrho,\alpha}(\mathsf{w})$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $R_{m_\varrho,\alpha}(\mathsf{r})$ | $\alpha$=0 | $\alpha$=0.1 | $\alpha$=0.2 | $\alpha$=0.28 | $\alpha$=0.3 | $\alpha$=0.4 | $\alpha$=0.5 |
| 5 | 20.00 | 16.00 | 14.40 | 12.80 | 11.52 | 11.20 | 9.60 | 8.00 |
| 6 | 16.67 | $\boxed{16.67}$ | 15.00 | 13.33 | 12.00 | 11.67 | 10.00 | 8.33 |
| 7 | 14.29 | 17.14 | $\boxed{15.43}$ | 13.71 | 12.34 | 12.00 | 10.29 | 8.57 |
| 8 | 12.50 | 17.50 | 15.75 | $\boxed{14.00}$ | $\boxed{12.60}$ | 12.25 | 10.50 | 8.75 |
| 9 | 11.11 | 17.78 | 16.00 | 14.22 | 12.80 | $\boxed{12.44}$ | 10.67 | 8.89 |
| 10 | 10.00 | 18.00 | 16.20 | 14.40 | 12.96 | 12.60 | $\boxed{10.80}$ | 9.00 |
| 11 | 9.09 | 18.18 | 16.36 | 14.55 | 13.09 | 12.73 | 10.91 | $\boxed{9.09}$ |
| 12 | 8.33 | 18.33 | 16.50 | 14.67 | 13.20 | 12.83 | 11.00 | 9.17 |

and, analogously,

$$\begin{aligned} R_{m_\varepsilon,\alpha}(\mathsf{by}) &= r_{m_\varepsilon,\alpha}(\{\mathsf{b},\mathsf{y}\})g^{\mathsf{by}}(\mathsf{b}) + r_{m_\varepsilon,\alpha}(\{\mathsf{r}\})g^{\mathsf{ry}}(\mathsf{r}) \\ &= \frac{2}{3}100. \end{aligned}$$

Thus, in this alternative way, we observe that, for positive $\alpha$, $R_{m_\varepsilon,\alpha}(\mathsf{by}) > R_{m_\varepsilon,\alpha}(\mathsf{ry})$, which is consistent with Ellsberg's observations.

## VI. SUMMARY & CONCLUSIONS

Generally, computational intelligence covers two streams of research. The main stream is connected with developing procedures yielding solutions to difficult problems, solutions that are as good as (or better than) those achievable by the best specialists. The second stream focuses on the roots of artificial intelligence that are reflected by the famous Turing test: to set up systems that exhibit the intelligent behavior of humans, i.e., behavior that is indistinguishable from that of humans.

In this paper, we describe a method belonging to the second stream of research, an approach for modeling decision-making behavior of humans in situations with ambiguous information. We start with an assumption that uncertainty is described by a bpa, which is interpreted in the framework of belief functions as generalized probability. The total subjective reward, which is to be maximized by the decision maker, is then computed in two steps: First, the bpa is transformed to an equivalent PMF. Second, the values of the PMF are reduced, using the subjective coefficient of ambiguity, into $r$-weights. These weights are then used to compute the total subjective reward defined as the weighted sum of values of a gain (reward) function. For the three examples considered in the paper, we show that the optimal decision with respect to such a subjective total reward function is consistent with observed experimental results. The *6-Color Example* illustrates the role of the subjective coefficient of ambiguity on the simplest possible example. The *One Red Ball Example* is used to show that the selection of probability transform matters. The *Ellsberg's example* is included because of two reasons. First, it is a classical, and well known, example. Second, it shows that the application of the proposed framework is not always trivial. Moreover, all the three examples reveal some issues that require further study.

In Section III we introduce three probability transforms. Using *One Red Ball Example* we show that, though the plausibility transform is the only one that is consistent with Dempster-Shafer theory of evidence, it is not suitable for decision making in the framework of the generalized probability theory interpretation of belief functions. In the three examples, the bpa's are such that the maximum entropy and pignistic transforms coincide. It is obvious that they do not always coincide. So, a natural question is: Which of these two transforms should be considered in case they are different? In a decision-making framework, Perez suggests considering a minimax barycenter as a representative of a convex set of PMFs [12]. Under what conditions does the barycenter coincide with the pignistic transform? If they do not always coincide, which one is better for decision making?

When studying the second experiment of the Ellsberg's example, we show that there are two ways of evaluating the total subjective reward yielding different results. We consider better the one that explains Ellsberg's observation. However, how does one recognize the best way of evaluating the total subjective reward in a more general situation?

As one of the anonymous reviewers noted, another direction of the further research is connected with the interpretation of the coefficient of ambiguity. This mathematical proposal should be complemented with experimental research studying to what extent the coefficient $\alpha$ is a characteristic of a decision subject, or, as quoted from the review: "Another experimental alternative would be to obtain the alpha value in one example, and then apply this value in the other examples for each subject and test the matching level of the decisions."


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