Mathematical Social Sciences 2 (1982) 299–315
North-Holland Publishing Company

THE BANZHAF POWER INDEX FOR POLITICAL GAMES*

Prakash P. SHENOY
School of Business, The University of Kansas, Lawrence, KS 66045, U.S.A.

Communicated by S. Brams
Received 6 April 1981
Revised 22 June 1981

The Banzhaf index of a voting game is a measure of a priori power of the voters. The model on which the index is based treats the voters symmetrically, i.e. the ideology, outlook, etc., of the voters influencing their voting behavior is ignored. Here we present a nonsymmetric generalization of the Banzhaf index in which the ideology of the voters affecting their voting behavior is taken into account. A model of ideologies and issues is presented. The conditions under which our model gives the Shapley–Shubik index (another index of a priori power of the voters) are given. Finally several examples are presented and some qualitative results are given for straight majority and pure bargaining games.

Key words: Banzhaf power index; Shapley–Shubik index; simple game.

1. Introduction. The potential for applications

This paper is concerned with measuring quantitatively the power of each individual in political voting systems as a function of the voting rule and the individual's ideology.

A voting system (including the voting rules) can be represented in the abstract by a 'simple game', invented by John von Neumann and Oskar Morgenstern in their 1944 classic, *Theory of Games and Economic Behavior* (1944). Speaking intuitively, a simple game is cooperative/competitive enterprise in which the only goal is to win and the only rule is a specification of which coalitions are empowered to do so. Most of the familiar examples of constitutional political machinery such as direct majority rule, weighted voting, bicameral legislatures, committees, etc., can be represented in the abstract by means of a simple game. Apart from the formal, constitutional rules that determine the outcome when the votes are counted, a simple game otherwise treats the players symmetrically, i.e., the ideology of the

* This research was supported in part by the University of Kansas General Research Fund and the School of Business Research Fund.

0165-4896/82/0000–0000/$02.75

players that affect their behavior in the voting system is not included in the definition of a simple game.

How much power does a participant have in a political voting system is an important question in political science. A precise definition and measurement of power will enable one to better understand the political processes. In general it is difficult to define the concept of power precisely.[1] In a simple game, if one defines the power as the ability to affect a change in the decision of the voting body, several mathematical power indices have been widely used in recent years. Of these, the two most widely accepted and used are the Shapley-Shubik power index (Shapley and Shubik, 1954) and the Banzhaf power index (Banzhaf, 1965, 1966, 1968a, 1968b). Both of these power indices are based on normative models which assume that the participants behave symmetrically, i.e., the ideology, habits, outlook, preference, etc., of the participants are disregarded. Such symmetry is desirable if the power index is to be used to design voting rules in a decision making body or to answer questions like: "Is this given decision rule fair – does it distribute power equitably?" However, as a descriptive tool to describe how members will act in a voting system, the assumption of symmetry is very often an invalid one.

Recently, Owen (1971) and Shapley (1977) have described a nonsymmetric generalization of the Shapley–Shubik power index for political games. Political games are simple games together with an ideological description of the players. In this paper we shall present a nonsymmetric generalization of the Banzhaf power index for political games.

Section 2 contains a brief introduction to simple games. The (symmetric) Banzhaf power index is described in Section 3. In Section 4 we present a model of ideologies and issues. The nonsymmetric Banzhaf power index is then defined in Section 5 in terms of this model. The conditions under which the Shapley–Shubik power index is obtained in our model are described in Section 6 along with a brief definition of the Shapley–Shubik power index. In Section 7 we illustrate our ideologically oriented power measure with some examples. We obtain the Banzhaf indices for three-person voting systems; for one- and two-dimensional ideological space models; for straight majority, unanimity, and veto player voting rules; and for various possible constellations of voter profiles. We also state some qualitative results for straight majority and pure bargaining games. In Appendix A we indicate a method to construct the constellation of voter profiles given the voting records of players on identifiable issues for a two-dimensional ideological space model.

2. Simple games

Simple games form a certain class of n-person cooperative games in which each

[1] See Nagel (1975) for a discussion of this problem.

coalition is either all powerful or completely ineffectual. Let $N=\{1,2,\dots,n\}$ denote the set of all *players* indexed by the first n natural numbers. Subsets of N are called *coalitions*. Let 2^N denote the set of all possible coalitions. A *simple game* can be represented by a pair (N, W), where W is the set of all winning coalitions such that:

$$\emptyset \notin W, \tag{1}$$

$$N \in W, \tag{2}$$

and for each $R \in 2^N$ and for each $T \in 2^N$,

$$(R \supset T \text{ and } T \in W) \Rightarrow R \in W. \tag{3}$$

A simple game is said to be *proper* iff the complement of every winning coalition is losing; i.e., in any partition of the players into coalitions, at most one coalition is winning. A winning coalition R is called *minimal winning* iff every proper subset of R is losing. (Note that all coalitions that are not winning are *losing* coalitions.) A simple game can also be represented by the pair (N, W^{m}) where W^{m} is the set of all minimal winning coalitions. Note that W is the set of all supersets of the elements in W^{m}.

If $k \notin \bigcup W^{\mathrm{m}}$, then player k is said to be a *dummy*. If $W^{\mathrm{m}}=\{\{i\}\}$, then player i is called a *dictator* and all other players are of course dummies. If $j \in \bigcap W^{\mathrm{m}} \neq \emptyset$, then player j is said to be a *veto player*.

A weighted majority game is a simple game that can be represented by the symbol

$$[q; a_1, a_2, \dots, a_n] \tag{4}$$

where $q \geq 0$ is called the *quota*, $a_i \geq 0$ is the *weight* associated with the ith player and $R \in W$ iff $\sum_{i \in R} a_i \geq q$. Note that the weighted majority game represented by (4) is proper if $q > (a_1 + \cdots + a_n)/2$.

Example 2.1. The most common of all simple games is the *straight majority* game M_n, n odd, in which

$$W^{\mathrm{m}} = \{R \in 2^N : |R| = (n+1)/2\}$$

where $|R|$ denotes the cardinality of coalition R. A weighted voting representation of the game M_n is

$$[(n+1)/2;\ 1, 1, \dots, 1].$$

Example 2.2. The *pure bargaining* game (or *unanimity* game) B_n is given by $W^{\mathrm{m}} = \{N\}$. A weighted majority representation of this game is

$$[n;\ 1, 1, \dots, 1].$$

Example 2.3. A three-person *veto player game* $V_3(1)$ is given by $W^{\mathrm{m}} = \{\{1,2\},\{1,3\}\}$.

In other words, the voting rule is straight majority with player 1 having veto power. A weighted majority representation of this game is

$$[3; 2, 1, 1].$$

See Shapley (1962) for a detailed description of simple games. Also Lucas (1976) presents several real life examples of organizations, committees, and legislatures modelled as simple games along with their Shapley–Shubik and Banzhaf power indices.

3. The Banzhaf power index

One way of looking at how a voting system distributes power among its members is to suppose that each bill will induce a probability p^i with which member i will vote 'aye' for the bill (and vote 'nay' with probability $1-p^i$). If member i is strongly for the bill, then p^i will be close to 1, if he is strongly against it, p^i will be close to 0, and if he is indifferent to it, p^i will be close to $\frac{1}{2}$.

For a measure of a priori power – by which we mean abstract power within the given voting system, not power with respect to any particular issue or goal that the system faces – we may well assume that each $p^i=\frac{1}{2}$, i.e. each voter i regards the others as random decision makers voting 'aye' or 'nay' (independently) at the toss of a coin. A player, say i, then asks: "What chance do I have of being able to tip the scales and decide the outcome?" Let us call a pair of coalitions of the form $(S-\{i\}, S\cup\{i\})$ a *swing for i* iff the former is losing and the latter is winning and let us denote by Y_i the subset of $N-\{i\}$ that votes 'aye'. Then i's chance of being decisive is just the probability that $(Y_i, Y_i\cup\{i\})$ is a swing for player i. This provides an index of the a priori power inherent in the given voting rule and is called the *Banzhaf index* and will be denoted here by β_i, $i\in N$.

Example 3.1. Consider the straight majority game described in Example 2.1. $(Y_i, Y_i\cup\{i\})$ is a swing for player i iff $|Y_i|=(n+1)/2-1=(n-1)/2$. *Hence*

$$\beta_i=\binom{n-1}{(n-1)/2}/2^{n-1}.$$

By symmetry all players have the same power. Hence if $n=3$, then each $\beta_i=\frac{1}{2}$; if $n=5$, then each $\beta_i=\frac{3}{8}$. Thus we see that although the relative power is the same in all such games, each player has less absolute power in a larger member voting body.

Example 3.2. Consider the pure bargaining game B_n described in Example 2.2. $(Y_i, Y_i\cup\{i\})$ is a swing for player i iff $Y_i=N-\{i\}$. Hence $\beta_i=1/2^{n-1}$. By symmetry all the players have the same power. If $n=3$, then each $\beta_i=\frac{1}{4}$; if $n=4$, then each $\beta_i=\frac{1}{8}$; if $n=5$, then each $\beta_i=\frac{1}{16}$, and so on. In a same size voting body, a member has more power if the voting rule is straight majority rule than if the voting rule is unanimity.

Example 3.3. Consider the 3-person veto player game $V_3(1)$ described in Example 2.3. $(Y_1, Y_1 \cup \{1\})$ is a swing for player 1 iff $Y_1 = \{2\}$ or $\{3\}$ or $\{2,3\}$. Hence $\beta_1 = \frac{3}{4}$. $(Y_2, Y_2 \cup \{2\})$ is a swing for player 2 iff $Y_2 = \{1\}$. Hence $\beta_2 = \frac{1}{4}$. By symmetry $\beta_3 = \frac{1}{4}$. Note that the veto power player has more power than the other players.

4. A model of ideologies and issues

This model is in many respects similar to the model proposed by Owen (1971) and Shapley (1977) though there are many differences.

We shall represent the voters by points in the finite-dimensional Euclidean space $\mathbb{R}^m$. We shall refer to this space as the *ideological space*. The linear structure of $\mathbb{R}^m$ will help us capture intuitive ideas like 'moderation' and 'extremism', e.g., by setting $m = 1$, we can think of $\mathbb{R}^1$ as a left to right spectrum of political ideology. Each dimension in this space represents political or ideological parameters, e.g., left/right (economic), left/right (international), urban/rural, free trade/protectionism, isolation/internationalism, economic growth/environment, big/small (government), big/small (business), etc. For reasons that will be obvious later, we shall restrict the ideological space to be the half-ball $B^m_{1/2}$ where

$$B^m_{1/2} = \{x \in R^m : (x_1^2 + x_2^2 + \cdots + x_m^2)^{1/2} \le \tfrac{1}{2}\}. \tag{5}$$

Each member i will be represented by a point x^i in the half-ball $B^m_{1/2}$. The political/ideological parameters of the members will determine his position. x^i will be called the *political profile* of voter i and the collection of voter profiles will be called a *constellation* and denoted by $C(N)$, i.e.,

$$C(N) = \{x^i \in B^m_{1/2} : i \in N\}. \tag{6}$$

If each member's position on each ideological parameter is known, the constellation of voter profiles can be determined as follows. For each parameter j, player i is placed in the interval $[-1, 1]$. Let Z^i_j represent voter i's position on parameter j. Then $-1 \le Z^i_j \le 1$, $Z^i_j = -1$ representing the left extreme, $Z^i_j = 1$ the right extreme, and $Z^i_j = 0$ a moderate position. A voter can thus be represented by a point in $[-1, 1]^m$. This space is then transformed to the half-ball $B^m_{1/2}$ by a continuous mapping f given by

$$[-1, 1]^m \xrightarrow{f} B^m_{1/2},$$

$$f(Z) = \begin{cases} Z/2d(Z) & \text{if } Z \ne (0, 0, \ldots, 0), \\ (0, 0, \ldots, 0) & \text{if } Z = (0, 0, \ldots, 0) \end{cases} \tag{7}$$

where

$$d(Z)=d(Z_1,\ldots,Z_m)=\sup\{y:\max_{j=1,\ldots,m} yZ_j/(Z_1^2+\cdots+Z_m^2)^{1/2}\leq 1\}$$

if $Z\neq(0,\ldots,0)$.

Then $C(N)$ is given by

$$C(N)=\{x^i\in B^m_{1/2}: x^i=f(Z^i),\ Z^i\in[-1,1]^m\}. \tag{8}$$

Intuitively speaking, the m-cube $[-1,1]^m$ is radially compressed to the half-ball $B^m_{1/2}$ (see Fig. 1 for the case $m=2$).

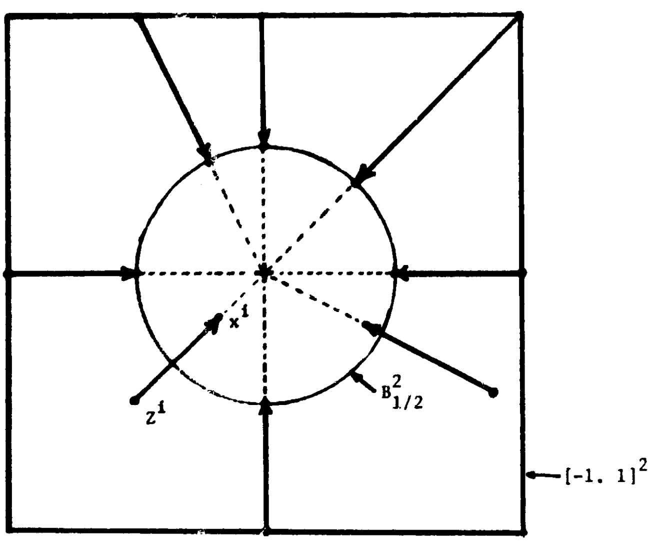


Fig. 1. The mapping $f:[-1,1]^2 \longrightarrow B^2_{1/2}$.

The assembly of voters will be presented with a series of bills or issues that arise out of some random process. Each bill or issue will consist of some combination of the parameters of the ideological space and will generate for each voter the probability with which he will vote 'aye' for the bill. We shall represent issues in our model as linear functions on $\mathbb{R}^m$; they will be used to generate the probability with which a voter is likely to vote 'aye' for the bill as follows:

$$p^i_\xi=\xi_1x^i_1+\xi_2x^i_2+\cdots+\xi_mx^i_m+1/2 \tag{9}$$

where $\xi=(\xi_1,\ldots,\xi_m)$ represents an issue vector of length one, i.e. $(\xi_1^2+\cdots+\xi_m^2)^{1/2}=1$, and $x^i\in B^m_{1/2}$. Intuitively speaking, voter i's enthusiasm for a bill represented by ξ is given by p^i_ξ which is proportional to the distance of his position x^i from the issue direction ξ. Thus issues in our model can be represented by points on the unit sphere S^m_1 where

$$S^m_1=\{\xi\in R^m: (\xi_1^2+\cdots+\xi_m^2)^{1/2}=1\}. \tag{10}$$

Although the political profiles x^i and political issues ξ are both m-dimensional, they should be regarded as belonging to different, dual spaces, not to the same space. Note that p^i_ξ satisfies $0 \leq p^i_\xi \leq 1$ for each $x^i \in B^m_{1/2}$ and each $\xi \in S^m_1$ and hence is a well-defined probability. Fig. 2 illustrates this graphically from $m=2$; the picture in higher dimensions would be entirely analogous. The long arrows indicate the directions associated with two typical issues ξ and ξ'. The corresponding probabilities are found by dropping perpendiculars to the shafts of the arrows. The probability varies uniformly from zero to one over the intersection of the shaft of the arrow with $B^m_{1/2}$ in the direction of the arrow. As the arrow turns, different probabilities are obtained; thus for a 180° turn, i.e., $\xi'' = -\xi$, we have $p^i_{\xi''} = 1 - p^i_\xi$.

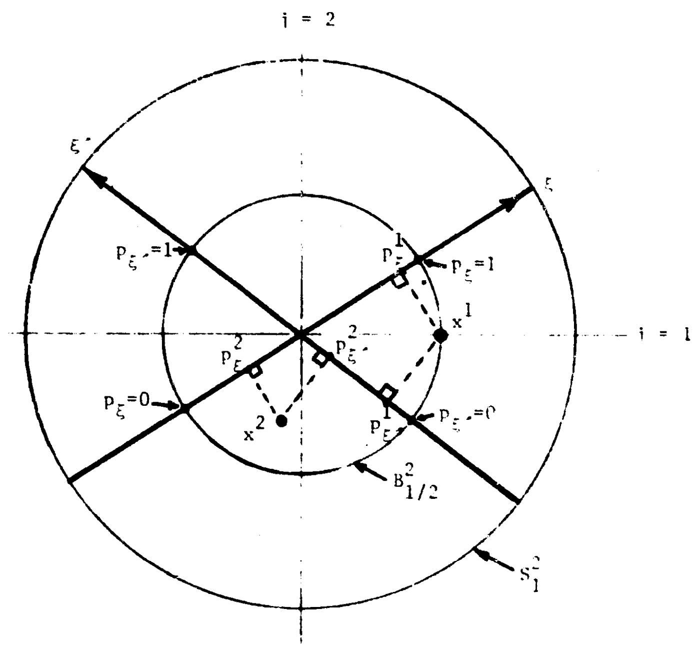


Fig. 2. Issues, voter profiles and the corresponding probabilities for each voter on each issue.

5. The Banzhaf index for political games

In order to define the Banzhaf index associated with a given voting rule and a given constellation $C(N)$ of voter profiles, we assume that all issue directions are equally likely, i.e., ξ is selected according to the uniform probability distribution on the unit sphere S^m_1. This amounts to saying that the 'political winds' blow across the ideological space in a perfectly random way, or in other words, all types of bills are equally likely to be introduced to an assembly of voters.[2]

[2] If in some particular context, this assumption is not satisfied, then we can replace the uniform probability distribution on S^m_1 by an appropriate probability distribution function.

We now consider the following question (see Straffin (1977)).

> *Question of effect on outcome:* What is the probability that player i's vote will make a difference in the outcome? That is, what is the probability that a bill which player i supports passes but would fail if player i changed his vote, or that a bill which player i opposes fails but would pass if player i changed his vote?

For some issue ξ (drawn at random from S_1^m according to the uniform distribution), the answer to this question of effect on outcome in our model is then defined to be player i's Banzhaf index and denoted by β_i.

Note that if all the players are located at the center of the half-ball $B_{1/2}^m$, the asymmetrical Banzhaf index will coincide with the symmetrical Banzhaf index defined in Section 3. This result is true even if the assumption of equally likely issue directions is replaced by another arbitrary probability distribution on S_1^m. We can ask the question: Under what assumptions will we get the Shapley-Shubik index in our model? This is examined in the next section.

6. The Shapley–Shubik power index

The Shapley–Shubik power index is another index of a priori power of players in a simple game. They assume that each bill or issue will rank the members in order to degree of their support – the most dedicated advocates first, the less dedicated supporters next, and so on, down to the most stubborn opponent at the end of the list. In any such ordering, one member will always play the role of the *pivot*: he in company with his more enthusiastic forerunners can just barely pass the bill. Assuming that all orderings of members will occur equally often (since we are measuring a priori power, not power with respect to a particular issue), we can then take the probability of being pivotal as a power index for each individual. This is known as the Shapley–Shubik index and we denote it here by Φ_i, $i \in N$.

In Straffin (1977), Straffin provides a characterization of the Shapley–Shubik index in terms of a probabilistic model that is similar in spirit to the one described in Section 3. Suppose each individual i votes 'aye' for a bill with probability p (p is the same for all individuals). If p is selected from $[0,1]$ by a uniform distribution, i.e., $p \sim U[0,1]$, the answer to the question of effect on outcome is given by the Shapley–Shubik index. Hence, one difference between the Shapley–Shubik and the Banzhaf index is that (in terms of the probability model) while the former assumes homogeneity among the members (same p for all members, $p \sim U[0,1]$), the latter assumes independence between each members (each $p^i \sim U[0,1]$).

The question we address is: Under what conditions do we get the Shapley–Shubik index in our model? For $m=2$, the answer to this question is given by the following theorem.

Theorem 6.1. *For $m=2$, if all the players are located at the point $(r,\theta)=(\frac{1}{2},\gamma)\in B^m_{1/2}$ (in polar coordinates) and the probability distribution of issues $\xi=(1,\theta)\in S^2_1$ is given by the density function*

$$f_\xi(\theta)=|\sin(\theta-\gamma)|/4 \quad \forall\theta,\ 0\leqq\theta\leqq 2\pi,$$

the answer to player i's question of effect on outcome in our model is given by the Shapley–Shubik index Φ_i for all $i\in N$.

Proof. Without loss of generality, let $\gamma=0$. For any issue direction $\xi=(1,\theta)\in S^2_1$ (in polar coordinates), the probability that player i (and all the rest of the players) will vote 'aye' is given by (see Fig. 3)

$$p^i_\theta=p_\theta=\tfrac{1}{2}\cos(\theta)+\tfrac{1}{2}.$$

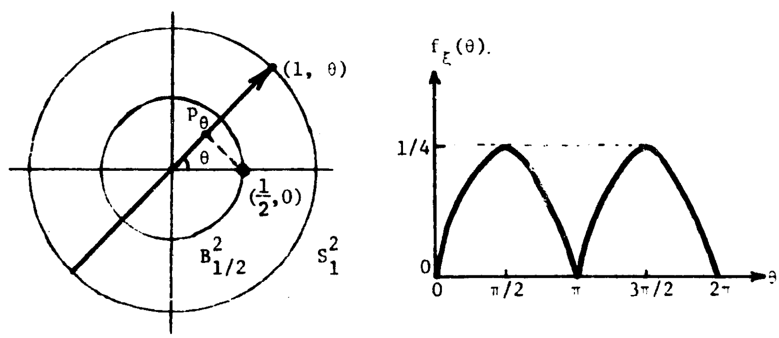


Fig. 3. The probability p_θ and the distribution $f_\xi(\theta)$.

The induced cumulative probability distribution of p_θ is given by:

$$F_{p_\theta}(z)=\text{Prob}[p_\theta\leqq z]=\text{Prob}[\tfrac{1}{2}\cos(\theta)+\tfrac{1}{2}\leqq z]=\text{Prob}[\cos(\theta)\leqq 2z-1]$$

$$=\text{Prob}[\cos^{-1}(2z-1)\leqq\theta\leqq 2\pi-\cos^{-1}(2z-1)]$$

$$=\int_{\cos^{-1}(2z-1)}^{2\pi-\cos^{-1}(2z-1)}\tfrac{1}{4}|\sin(\theta)|\,d\theta=\int_{\cos^{-1}(2z-1)}^{\pi}\tfrac{1}{4}\sin(\theta)d\theta-\int_{\pi}^{2\pi-\cos^{-1}(2z-1)}\tfrac{1}{4}\sin(\theta)d\theta$$

$$=\tfrac{1}{2}z+\tfrac{1}{2}z=z.$$

Hence, $p_\theta\sim U[0,1]$ and from our remarks earlier it follows that the answer to player i's question of effect on outcome is given by the Shapley–Shubik index Φ_i.

For higher dimensions, analogues of this result are possible. We will sketch the outline and skip the details. The players will have to be located together at any point on the boundary of $B^m_{1/2}$. The probability distribution of issue directions ξ should be such that the induced probability distribution of p_ξ is $U[0,1]$. Then using the result due to Straffin (1977), the answer to player i's question of effect on outcome will be given by the Shapley–Shubik index.

7. Some examples and qualitative results

We shall now illustrate our ideologically oriented power measure with some examples.

First let $m=1$. Then $B^1_{1/2}=[-\frac{1}{2},\frac{1}{2}]$ and $S^1_1=\{-1,1\}$, i.e., the ideology space is an interval from $-\frac{1}{2}$ to $\frac{1}{2}$ and there are only two issue directions $\xi=-1$ (left) and $\xi=1$ (right). Consider the case of three voters labelled A, B and C located at x^A, x^B and x^C respectively in $[-\frac{1}{2},\frac{1}{2}]$. To compute the Banzhaf index of the players, let $\psi^i(\xi)$ in the probability that player i's vote will make a difference on issue ξ. Then

$$\beta_i=\int_{\xi}\psi^i(\xi)\mathrm{d}F_{\xi} \tag{11}$$

where F_{ξ} is the uniform cumulative probability distribution function on S^m_1.

Under straight majority voting rule,

$$\psi^i(\xi)=p^j_{\xi}(1-p^k_{\xi})+(1-p^j_{\xi})p^k_{\xi} \quad \text{for } i,j,k=A,B,C,\ i\neq j,\ j\neq k,\ k\neq i;$$

and $p^j_{\xi}=x^j\xi+\frac{1}{2}$ for $j=A,B,C$. Hence

$$\beta=(\beta_A,\beta_B,\beta_C)=(\tfrac{1}{2}-2x^Bx^C,\ \tfrac{1}{2}-2x^Ax^C,\ \tfrac{1}{2}-2x^Ax^B).$$

See Table 1 for a list of the Banzhaf indices for various possible constellations of voter profiles. In that table, we also list the relative Banzhaf power index (also called the *normalized Banzhaf index*) for each player denoted by $\bar{\beta}_i$, i.e.,

$$\bar{\beta}_i=\beta_i/\sum_{i\in N}\beta_i \quad \text{if } \sum_{i\in N}\beta_i\neq 0. \tag{12}$$

Table 1
The Banzhaf indices of 3-person straight majority games M_3 for various possible constellations of voter profiles, $m=1$

#	Constellation	β	$\bar{\beta}$	$\sum_{i\in N}\beta_i$
1	|—— ABC ——|	(1/2, 1/2, 1/2)	(1/3, 1/3, 1/3)	1.5
2	|A —— BC ——|	(1/2, 1/2, 1/2)	(1/3, 1/3, 1/3)	1.5
3	|—— A —— BC|	(0, 1/2, 1/2)	(0, 1/2, 1/2)	1
4	|A —— B —— C|	(1/2, 1, 1/2)	(1/4, 1/2, 1/4)	2
5	|A —— BC|	(0, 1, 1)	(0, 1/2, 1/2)	2
6	|—— ABC|	(0, 0, 0)	(0, 0, 0)	0

Also the total absolute power $\sum_{i\in N}\beta_i$ is listed in the last column.

For the unanimity voting rule,

$$\psi^i(\xi)=p^j_{\xi}p^k_{\xi} \quad \text{for } i,j,k=A,B,C,\ i\neq j,\ j\neq k,\ k\neq i.$$

Hence,

$$\beta=(\beta_A,\beta_B,\beta_C)=(\tfrac{1}{4}+x^Bx^C,\ \tfrac{1}{4}+x^Ax^C,\ \tfrac{1}{4}+x^Ax^B).$$

See Table 2 for a list of the Banzhaf indices (both absolute and relative) for the various possible constellations of voter profiles.

Table 2
The Banzhaf indices of 3-person bargaining games B_3 for various possible constellations of voter profiles, $m=1$.

#	Constellation	β	β	$\sum_{i\in N}\beta_i$
1	|—— ABC ——|	(1/4, 1/4, 1/4)	(1/3, 1/3, 1/3)	0.75
2	|A —— BC ——|	(1/4, 1/4, 1/4)	(1/3, 1/3, 1/3)	0.75
3	|—— A —— BC|	(1/2, 1/4, 1/4)	(1/2, 1/4, 1/4)	1
4	|A —— B —— C|	(1/4, 0, 1/4)	(1/2, 0, 1/2)	0.5
5	|A ——— BC|	(1/2, 0, 0)	(1, 0, 0)	0.5
6	|——— ABC|	(1/2, 1/2, 1/2)	(1/3, 1/3, 1/3)	1.5

For the veto power player game $V_3(A)$ (player A has the veto power),

$$\psi^A(\xi)=p_\xi^B(1-p_\xi^C)+(1-p_\xi^B)p_\xi^C+p_\xi^B p_\xi^C,$$

$$\psi^B(\xi)=p_\xi^A(1-p_\xi^C), \quad \text{and} \quad \psi^C(\xi)=p_\xi^A(1-p_\xi^B).$$

Hence,

$$\beta=(\beta_A,\beta_B,\beta_C)=(\tfrac{3}{4}-x^Bx^C,\ \tfrac{1}{4}-x^Ax^C,\ \tfrac{1}{4}-x^Ax^B).$$

See Table 3 for a list of the Banzhaf indices (both absolute and relative) for the various possible constellations of voter profiles.

Table 3
The Banzhaf indices of 3-person veto player game $V_3(A)$ for various possible constellations of voter profiles, $m=1$

#	Constellation	β	β	$\sum_{i\in N}\beta_i$
1	|—— ABC ——|	(3/4, 1/4, 1/4)	(3/5, 1/5, 1/5)	1.25
2	|A —— BC ——|	(3/4, 1/4, 1/4)	(3/5, 1/5, 1/5)	1.25
3	|—— A —— BC|	(1/2, 1/4, 1/4)	(1/2, 1/4, 1/4)	1
4	|A —— B —— C|	(3/4, 1/2, 1/4)	(1/2, 1/3, 1/6)	1.5
5	|A ——— BC|	(1/2, 1/2, 1/2)	(1/3, 1/3, 1/3)	1.5
6	|——— ABC|	(1/2, 0, 0)	(1, 0, 0)	0.5

By studying the Banzhaf indices for the straight majority game M_3 and the pure bargaining game B_3 in Tables 1 and 2, we can derive some qualitative results. For $m=1$ we defined $x_j^i=-\frac{1}{2}$ as an (absolute) left extreme position, $x_j^i=0$ as an (absolute) moderate position, and $x_j^i=\frac{1}{2}$ as an (absolute) right extreme position on parameter j. We will define relative left and right extreme position and relative moderation as follows. We say player i occupies a *relative left extreme* position on parameter j iff $x_j^i < x_j^k$ for each $k \in N - \{i\}$. Similarly, we say player i occupies a *relative right extreme* position on parameter j iff $x_j^i > x_j^k$ for each $k \in N - \{i\}$. Also we say player i occupies a *relative moderate* position on parameter j iff he occupies neither a relative left extreme position nor a relative right extreme position (on parameter j). Thus (referring to Tables 1–3), player A occupies the relative moderate position in constellations 1 and 6, and the relative left extreme position in constellations 2–5; player B occupies the relative moderate position in constellations 1–6; player C occupies the relative moderate positions in constellations 1–3 and 5–6, and the relative right extreme position in constellation 4.

Observing the Banzhaf indices of the players in Table 1 for the straight majority voting rule, note that in every constellation player B has the highest Banzhaf index compared to the other two players. Thus we can conclude that *relative moderate position is advantageous in straight majority games.*

Observing the Banzhaf indices of the players in Table 2 for the unanimity voting rule, note that a relative extremist never has less power than a relative moderate player. Thus we can conclude that *relative extremism is beneficial in pure bargaining games.*

Next let $m = 2$ and consider again the case of three players labelled A, B and C with profiles x^A, x^B and x^C respectively. Using (11) it can be easily shown that for the straight majority game M_3,

$$\begin{aligned}\beta &= (\beta_A, \beta_B, \beta_C)\\ &= (\tfrac{1}{2} - (x_1^B x_1^C + x_2^B x_2^C), \tfrac{1}{2} - (x_1^A x_1^C + x_2^A x_2^C), \tfrac{1}{2} - (x_1^A x_1^B + x_2^A x_2^B)).\end{aligned}$$

For the unanimity voting rule B_3, it can be easily shown that

$$\begin{aligned}\beta &= (\beta_A, \beta_B, \beta_C)\\ &= (\tfrac{1}{4} + \tfrac{1}{2}(x_1^B x_1^C + x_2^B x_2^C), \tfrac{1}{4} + \tfrac{1}{2}(x_1^A x_1^C + x_2^A x_2^C), \tfrac{1}{4} + \tfrac{1}{2}(x_1^A x_1^B + x_2^A x_2^B)).\end{aligned}$$

For the veto player game $V_3(A)$, it can be easily shown that

$$\begin{aligned}\beta &= (\beta_A, \beta_B, \beta_C)\\ &= (\tfrac{3}{4} - \tfrac{1}{2}(x_1^B x_1^C + x_2^B x_2^C), \tfrac{1}{4} - \tfrac{1}{2}(x_1^A x_1^C + x_2^A x_2^C), \tfrac{1}{4} - \tfrac{1}{2}(x_1^A x_1^B + x_2^A x_2^B)).\end{aligned}$$

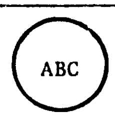
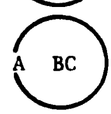
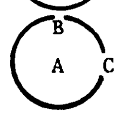
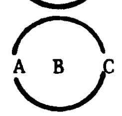
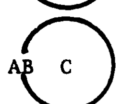
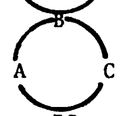
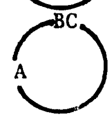
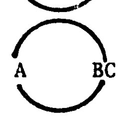
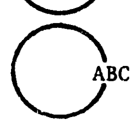
See Tables 4, 5 and 6 for a list of the Banzhaf indices for the games M_3, B_3 and $V_3(A)$ respectively, for some possible constellations of voter profiles.

Since we have two parameters, we shall define extremism and relative moderation as follows. We say player i is a *relative left extremist* if $x_1^i \leq x_1^k$ for each $k \in N - \{i\}$, $x_2^i \leq x_2^k$ for each $k \in N - \{i\}$, and at least one of the above two inequalities is a strict inequality. Relative right extremism is defined similarly as follows. We say player i is a *relative right extremist* if $x_1^i \geq x_1^k$ for each $k \in N - \{i\}$, $x_2^i \geq x_2^k$ for each $k \in N - \{i\}$, and at least one of the above two inequalities is a strict inequality. Also, we say player i is a *relative moderate* if he is neither a relative left extremist nor a relative right extremist. Thus referring to Tables 4, 5 and 6, we see that player A is a relative moderate in constellations 1, 3, 5 and 9, and is a relative left extremist in constellations 2, 4, 6–8; player B is a relative moderate in all constellations; player C is a relative moderate in constellations 1–3, 6–9, and is a relative right extremist in constellations 4 and 5.

Observing the Banzhaf indices of the players in Table 4, we observe that in every constellation, player B (who is a relative moderate in all the constellations) has the highest Banzhaf index compared to the relative extremists in the constellation. Thus, we reinforce our earlier conclusion that relative moderation is advantageous in straight majority games.

Observing the Banzhaf indices of the players in Table 5, we observe that the Banzhaf indices for player A in constellations 2, 4, 6–8 (in which he is a relative left extremist) and the Banzhaf indices of player C in constellations 4 and 5 (in which he

Table 4
The Banzhaf indices of 3-person straight majority games M_3 for various constellations of voter profiles, $m = 2$

#	Constellation	β	$\bar{\beta}$	$\sum_{i \in N} \beta_i$
1	 ABC	(1/2, 1/2, 1/2)	(1/3, 1/3, 1/3)	1.5
2	 A BC	(1/2, 1/2, 1/2)	(1/3, 1/3, 1/3)	1.5
3	 B A C	(1/2, 1/2, 1/2)	(1/3, 1/3, 1/3)	1.5
4	 A B C	(1/2, 3/4, 1/2)	(2/7, 3/7, 2/7)	1.75
5	 AB C	(1/2, 1/2, 1/4)	(2/5, 2/5, 1/5)	1.25
6	 B A C	(1/2, 3/4, 1/2)	(2/7, 3/7, 2/7)	1.75
7	 BC A	(1/4, 1/2, 1/2)	(1/5, 2/5, 2/5)	1.25
8	 A BC	(1/4, 3/4, 3/4)	(1/7, 3/7, 3/7)	1.75
9	 ABC	(1/4, 1/4, 1/4)	(1/3, 1/3, 1/3)	0.75

is a relative right extremist) are greater than the Banzhaf indices of the relative moderate players in those constellations. Thus again our earlier conclusion that relative extremism is beneficial in pure bargaining games is reinforced.

The two conclusions we have stated are certainly no revelations to those playing these games in real-life political institutions. The validity of these results underscores the validity of our model and also illustrates some of the usefulness of our model in deriving results of this type. A real test of our model will be in using it in a real-life institutional setting and verifying that the power indices which our model indicates are empirically valid. In a step towards that goal, we indicate a method of constructing the constellation of voter profiles if empirical voting data on identifiable issues are available. This is discussed in Appendix A.

Table 5
The Banzhaf indices of 3-person pure bargaining games B_3 for various possible constellations of voter profiles, $m=2$

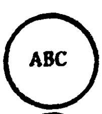
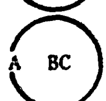
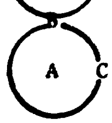
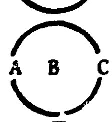
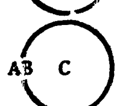
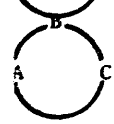
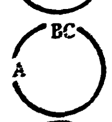
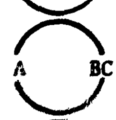
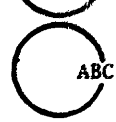
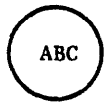
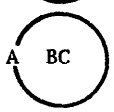
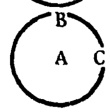
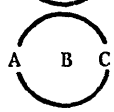
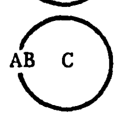
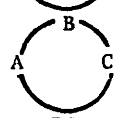
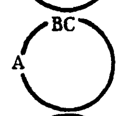
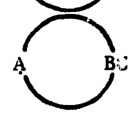
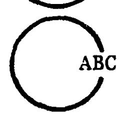
#	Constellation	β	$\bar{\beta}$	$\sum_{i\in N} \beta_i$
1		(1/4, 1/4, 1/4)	(1/3, 1/3, 1/3)	0.75
2		(1/4, 1/4, 1/4)	(1/3, 1/3, 1/3)	0.75
3		(1/4, 1/4, 1/4)	(1/3, 1/3, 1/3)	0.75
4		(1/4, 1/8, 1/4)	(2/5, 1/5, 2/5)	0.625
5		(1/4, 1/4, 3/8)	(2/7, 2/7, 3/7)	0.875
6		(1/4, 1/8, 1/4)	(2/5, 1/5, 2/5)	0.625
7		(3/8, 1/4, 1/4)	(3/7, 2/7, 2/7)	0.875
8		(3/8, 1/8, 1/8)	(3/5, 1/5, 1/5)	0.625
9		(3/8, 3/8, 3/8)	(1/3, 1/3, 1/3)	1.125

Table 6
The Banzhaf indices of 3-person veto player games $V_3(A)$ for various possible constellations of voter profiles, $m=2$

#	Constellation	β	$\bar{\beta}$	$\sum_{i \in N} \beta_i$
1		(3/4, 1/4, 1/4)	(3/5, 1/5, 1/5)	1.25
2		(3/4, 1/4, 1/4)	(3/5, 1/5, 1/5)	1.25
3		(3/4, 1/4, 1/4)	(3/5, 1/5, 1/5)	1.25
4		(3/4, 3/8, 1/4)	(6/11,3/11,2/11)	1.375
5		(3/4, 1/4, 1/8)	(2/3, 2/9, 1/9)	1.125
6		(3/4, 3/8, 1/4)	(6/11,3/11,2/11)	1.375
7		(5/8, 1/4, 1/4)	(5/9, 2/9, 2/9)	1.125
8		(5/8, 3/8, 3/8)	(5/11,3/11,3/11)	1.375
9		(5/8, 1/8, 1/8)	(5/7, 1/7, 1/7)	0.875

Acknowledgment

This research was inspired by Dr. Lloyd Shapley and his research on political games, and was partly done during the author's visit to the Rand Corporation during the summer of '79 (made possible by Dr. Shapley). The research benefitted immensely from the almost daily discussions the author had with Dr. Shapley during the visit. Of course, the author alone is responsible for any shortcomings, etc., of this paper.

Appendix A. Construction of the constellation of voter profiles given the voting records of the players on identifiable issues, for $m=2$

The problem we consider here is the question of constructing the constellation of voter profiles given the players' voting record on issues whose directions are known. We shall assume that we have information regarding the dimensions of the ideological space and each issue can thus be represented by a point in the dual space.

Suppose player i's true position is at (r_i, θ_i) in $B^2_{1/2}$ (in polar coordinates). Consider an arbitrary issue direction $(1, \theta)$. The probability that player i will vote 'aye' on this issue will be given by $p^i_\theta = r_i\cos(\theta_i)\cos(\theta) + r_i\sin(\theta_i)\sin(\theta) + \frac{1}{2}$. If sufficient data are available (in the form of voting patterns) for this issue direction, this probability can be estimated and represented by the point

$$(p^i_\theta - 1/2, \theta) = (r_i\cos(\theta_i)\cos(\theta) + r_i\sin(\theta_i)\sin(\theta), \theta)$$

(in polar coordinates, in $B^2_{1/2}$).

If all the issue directions are equally likely and we take the average of all the x_1-(Cartesian) coordinates and the x_2-(Cartesian) coordinates of the points representing the probabilities p^i_θ, we get

$$(1/2\pi)\int_0^{2\pi}(r_i\cos(\theta_i)\cos(\theta) + r_i\sin(\theta_i)\sin(\theta))\cos(\theta)\mathrm{d}\theta = \tfrac{1}{2}r_i\cos(\theta_i),$$

$$(1/2\pi)\int_0^{2\pi}(r_i\cos(\theta_i)\cos(\theta) + r_i\sin(\theta_i)\sin(\theta))\sin(\theta)\mathrm{d}\theta = \tfrac{1}{2}r_i\sin(\theta_i),$$

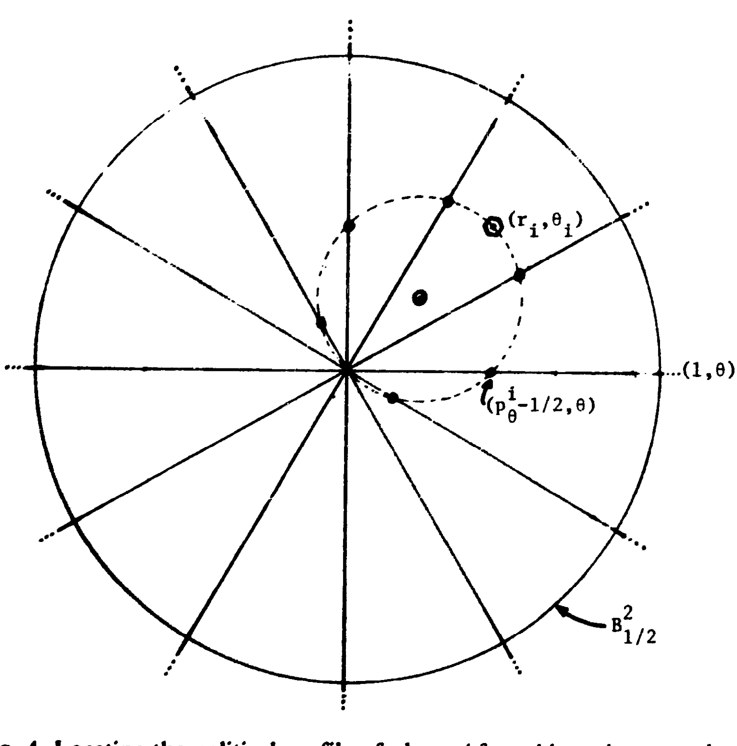


Fig. 4. Locating the political profile of player i from his voting record.

Hence, multiplying the average of all the points representing the probabilities p_θ^i by 2, we can locate the true position of player i in the ideological space $B_{1/2}^2$ (see Fig. A.1).

References

J.F. Banzhaf III, Weighted voting doesn't work: A mathematical analysis, Rutgers Law Rev. 19 (1965) 317–343.

J.F. Banzhaf III, Multi-member electoral districts – Do they violate the 'One Man, One Vote' principle?, Yale Law J. 75 (1966) 1309–1338.

J.F. Banzhaf III, One man, 3.312 votes: A mathematical analysis of the electoral college, Villanova Law Rev. 13 (1968a) 304–332.

J.F. Banzhaf III, One man, ? votes: Mathematical analysis of political consequences and judicial choices, George Washington Law Rev. 36 (1968b) 808–823.

W.F. Lucas, Measuring power in weighted voting systems, in: M.D. Thompson, ed., Case Studies in Applied Mathematics (Math. Assoc. Amer., Washington, DC, 1976).

J. Nagel, The Descriptive Analysis of Power (Yale University Press, New Haven, 1975).

J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (Princeton University Press, Princeton, 3rd ed., 1953).

G. Owen, Political games, Naval Res. Logistics Quart. 18 (1971) 345–355.

L.S. Shapley, Simple games: An outline of the descriptive theory, Behav. Sci. 7 (1962) 59–66.

L.S. Shapley, A comparison of power indices and a nonsymmetric generalization, Rand Paper P-5872, The Rand Corporation, Santa Monica, CA, 1977.

L.S. Shapley and M. Shubik, A method for evaluating the distribution of power in a committee system, Amer. Political Sci. Rev. 48 (1954) 787–792.

P.D. Straffin, Homogeneity, independence and power indices, Public Choice 30 (1977) 107–118.