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QUALITATIVE MARKOV NETWORKS

by

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I. Introduction

A lot of interest on the subject of management of uncertainty in expert systems has been devoted to propagation of belief functions and probabilities in networks (see, for example, Gordon and Shortliffe (1985), Shafer and Logan (1985), Pearl (1985)). In Shafer, Shenoy and Mellouli (1986), a scheme for propagating belief functions in "qualitative Markov trees" is presented. This scheme is a generalization of both the Shafer and Logan's scheme for hierarchical trees and Pearl's scheme for Bayesian causal trees (Shenoy and Shafer (1986)). In this paper, we concentrate on qualitative Markov trees and their properties. We start with a definition of conditional qualitative independence (q-independence) for partitions. We treat partitions as qualitative descriptions of belief functions and random variables. Using the concept of conditional q-independence, we define a qualitative Markov (q-Markov) network analogous to a Markov network (see, for example, Griffeath (1976), Darroch, Lauritzen and Speed (1980)). We then introduce the concept of a K-pattern for a collection of partitions (see, e.g., Kong (1986)) and prove that a tree of partitions is a q-Markov tree if and only if its edges form a K-pattern. For more general networks, we show that if a set of complete subsets of vertices of the network form a K-pattern, then the network is q-Markov. The paper ends with some comments on Shafer, Shenoy and Mellouli's (1986) propagation scheme for belief functions in q-Markov trees.

II. Qualitative Independence for Partitions

In this paper, we will be concerned with a finite indexed collection of partitions $\{\mathfrak{P}_j \mid j \in J\}$ of a finite nonempty set $\Omega = \{\omega_i \mid i \in I\}$. Such partitions can serve as qualitative descriptions of random variables or belief functions. To a random variable $X: \Omega \to R$, we associate the partition $\mathfrak{P}_X = \{P \in 2^{\Omega} \mid P = X^{-1}(a) \text{ for some } a \in X(\Omega)\}$ and to a belief function on a frame of discernment Ω, we associate the partition generated by taking intersections of the belief function's focal elements (Shafer (1976)).

Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be two distinct partitions. We say that $\mathfrak{P}_1$ ***is coarser than*** $\mathfrak{P}_2$ (or equivalently that $\mathfrak{P}_2$ ***is finer than*** $\mathfrak{P}_1$), written as $\mathfrak{P}_1 < \mathfrak{P}_2$, if for each $P_2 \in \mathfrak{P}_2$, there exists $P_1 \in \mathfrak{P}_1$ such that $P_1 \supseteq P_2$. We call $\mathfrak{P}_1$ ***a coarsening of*** $\mathfrak{P}_2$ and $\mathfrak{P}_2$ ***a refinement of*** $\mathfrak{P}_1$. We write $\mathfrak{P}_1 \leq \mathfrak{P}_2$ to indicate that $\mathfrak{P}_1$ is ***coarser than or equal to*** $\mathfrak{P}_2$. The relation $\leq$ is a partial order and the set of all partitions is a lattice with respect to this partial order (Birkhoff (1967)).The ***coarsest common refinement of*** $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ or the least upper bound of $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ with respect to $\leq$, denoted by $\vee\{\mathfrak{P}_j \mid j = 1, \ldots, n\}$ or by $\mathfrak{P}_1 \vee \ldots \vee \mathfrak{P}_n$, is the partition $\{ P_1 \cap \ldots \cap P_n \mid P_j \in \mathfrak{P}_j, \text{ for } j = 1, \ldots, n, \text{ and } P_1 \cap \ldots \cap P_n \neq \varnothing \}$.

We say that $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ are ***qualitatively independent*** (***q-independent***), written as $[\mathfrak{P}_1, \ldots, \mathfrak{P}_n] \dashv$, if for any $P_j \in \mathfrak{P}_j$ for $j = 1, \ldots, n$, we have $P_1 \cap \ldots \cap P_n \neq \varnothing$. Furthermore, we say that $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ are ***conditionally q-independent given*** $\mathfrak{P}$, written as $[\mathfrak{P}_1, \ldots, \mathfrak{P}_n] \dashv \mathfrak{P}$, if whenever we select $P \in \mathfrak{P}$, $P_i \in \mathfrak{P}_i$ for $i = 1, \ldots, n$ such that $P \cap P_i \neq \varnothing$ for $i = 1, \ldots, n$, then $P \cap P_1 \cap \ldots \cap P_n \neq \varnothing$. Notice that stochastic conditional independence implies qualitative conditional independence. If $\Omega = \{ \omega_i \mid i \in I\}$ represents a finite sample space, and $Pr: 2^{\Omega} \to [0, 1]$ represent a probability distribution on Ω such that $Pr(\{\omega_i\}) > 0$ for all $i \in I$, and X, Y, Z are random variables such that X and Y are conditionally independent given Z (written as $X \perp Y \mid Z$), then $[\mathfrak{P}_X, \mathfrak{P}_Y] \dashv \mathfrak{P}_Z$ where $\mathfrak{P}_X$, $\mathfrak{P}_Y$ and $\mathfrak{P}_Z$ are the partitions associated with X, Y, and Z, respectively.

III. Q-Markov Networks

We now consider networks where the nodes represents partitions and the edges represent certain conditional q-independence restrictions on the partitions . Consider a ***network***

(J, E), where J is a finite set of partitions thought of as the ***vertices*** of the network, and $E \subseteq J \times J$ is a set of unordered pairs of distinct elements of J, thought of as the ***edges*** of the network. We say that $i \in J$ and $j \in J$ are ***adjacent*** or ***neighbors*** if $(i, j) \in E$. If $J_1 \subseteq J$, the ***boundary*** of J_1, written as ∂J_1, is the set of vertices in $J \setminus J_1$ that are adjacent to some vertex in J_1. The ***closure*** of J_1 is $J_1 \cup \partial J_1$ and is denoted by $\overline{J_1}$. A ***complete subset*** (of vertices) is a subset $J_1 \subseteq J$ where all elements are mutual neighbors. A ***clique*** is a maximal (w.r.t. inclusion) complete subset.

A ***q-Markov network for*** $\{\mathfrak{P}_j \mid j \in J\}$ is a network (J, E) such that given any three mutually disjoint subsets J_1, J_2, and J_3 of J, if J_1 and J_2 are ***separated*** by J_3 (in the sense that any path from a vertex in J_1 to a vertex in J_2 goes via some vertex in J_3), then

$$[\vee\{\mathfrak{P}_j \mid j \in J_1\}, \vee\{\mathfrak{P}_j \mid j \in J_2\}] \dashv \vee\{\mathfrak{P}_3 \mid j \in J_3\}.$$

If (J, E) is a q-markov network for $\{\mathfrak{P}_j \mid j \in J\}$ and the network (J, E) is a tree, then we say that (J, E) is a ***q-markov tree for*** $\{\mathfrak{P}_j \mid j \in J\}$.

<u>Theorem</u> (J, E) is q-Markov for $\{\mathfrak{P}_j \mid j \in J\}$ if and only if for all $J_1 \subseteq J$,

$$[\vee\{\mathfrak{P}_j \mid j \in J_1\}, \vee\{\mathfrak{P}_j \mid j \in J \setminus \overline{J_1}\}] \dashv \vee\{\mathfrak{P}_j \mid j \in \partial J_1\}.$$

Let $\{\mathfrak{P}_i \mid i \in J\}$ be an indexed collection of partitions. Let $E \subseteq 2^J$. E is said to be a *K-pattern* for $\{\mathfrak{P}_i \mid i \in J\}$ if whenever we select an element P_i from $\mathfrak{P}_i$ for each $i \in J$ such that $\cap\{P_i \mid i \in I\} \neq \emptyset$ for all $I \in E$, then $\cap\{P_i \mid i \in J\} \neq \emptyset$. Notice that if the elements of $\{\mathfrak{P}_i \mid i \in J\}$ are q-independent, then every $E \subseteq 2^J$ (including the empty set) is a K-pattern for $\{\mathfrak{P}_i \mid i \in J\}$. Also the singleton {J} is always a K- pattern for $\{\mathfrak{P}_i \mid i \in J\}$. It can be shown that if

E is a K-pattern for $\{\mathfrak{P}_i \mid i \in J\}$ then (J, E) is a q-Markov network. (In fact, if a set of complete subsets of vertices in (J, E) is a K-pattern for $\{\mathfrak{P}_i \mid i \in J\}$, then (J, E) is a q-Markov network for $\{\mathfrak{P}_i \mid i \in J\}$.The converse of this result is not valid for networks in general, i.e., neither the set of all edges nor the set of all complete subsets of vertices of a q-Markov network necessarily form a K-pattern.

IV. Q-Markov Trees

We now turn our attention to the case of q-Markov trees . Two characterizations of q-Markov trees are as follows.

Theorem Let $\{\mathfrak{P}_j \mid j \in J\}$ be a finite collection of partitions and let (J, E) be a tree. Then (J, E) is q-markov for $\{\mathfrak{P}_j \mid j \in J\}$ if and only if for every $j \in J$, $[\vee\{\mathfrak{P}_i \mid i \in \alpha_1(j)\}, \ldots, \vee\{P_i \mid i \in \alpha_k(j)\}] \dashv \mathfrak{P}_j$ where $\alpha_1(j), \ldots, \alpha_k(j)$ are separated by $\{j\}$ in the tree (J, E).

Theorem (J, E) is a q-Markov tree for $\{\mathfrak{P}_i \mid i \in J\}$if and only if E is a K-pattern for $\{\mathfrak{P}_i \mid i \in J\}$.

V. Conclusion

In this paper, we focused on characterizing q-Markov trees. Q-Markov trees are in fact the kind of trees for which we have developed a scheme for propagating belief functions with only "local computations" (see Shafer, Shenoy and Mellouli (1986) and Shenoy and Shafer (1986)). The local computation aspect of the scheme results in a reduction in the computational complexity associated with Dempster's rule of combination for belief functions and also makes possible an implementation in parallel that further reduces the time required for the computation. This computation scheme is a generalization of both Shafer and Logan's scheme for hierarchical trees and Pearl's scheme for Bayesian causal trees.

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