International Journal of Approximate Reasoning 141 (2022) 1–4

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Editorial

Probability and statistics: Foundations and history. Special Issue in honor of Glenn Shafer

John Aldrich [a], A. Philip Dawid [b], Thierry Denoeux [c,*], Prakash P. Shenoy [d], Vladimir Vovk [e]

[a] *Southampton Statistical Sciences Research Institute, University of Southampton, Highfield, Southampton SO17 1BJ, UK*
[b] *Statistical Laboratory, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK*
[c] *Université de technologie de Compiègne, UMR CNRS 7253 Heudiasyc, Rue Roger Couttolenc, CS 60319, 60203 Compiègne Cedex, France*
[d] *University of Kansas, School of Business, 1654 Naismith Drive, Lawrence, KS 66045, USA*
[e] *Department of Computer Science, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK*

ARTICLE INFO

Keywords:
Glenn Shafer
Foundations of statistics
Foundations of probability
Conformal prediction
History of probability
Causality

Glenn Shafer celebrated his 75th birthday in November 2021. This Special Issue of the *International Journal of Approximate Reasoning* in his honor, and on occasion of this event, is devoted to the main areas of his research.

The Special Issue starts with Glenn's short biography, continues with original research contributions by his colleagues and friends, and concludes with three papers by Glenn. In this introduction to the Special Issue we will try to put its contents, especially the original research contributions, in a wider context of Glenn's research.

Glenn's main research interest has been in the foundations of probability and statistics. While he was still a PhD student he became interested in Arthur Dempster's novel ideas in statistical inference, developing them into what later became known as Dempster-Shafer theory [11,14]. This Special Issue does not contain any papers devoted specifically to Dempster-Shafer theory, even though it is Glenn's oldest research interest and he remains keenly interested in it. This area was covered extensively in another Special Issue, "40 years of Research on Dempster-Shafer Theory" [6], of the *International Journal of Approximate Reasoning*, which also includes a fascinating intellectual autobiography [20].

Dempster-Shafer theory was Glenn's first foray into the area of imprecise probabilities. Next we will discuss game-theoretic probability and conformal prediction, which are also in this class.

Betting and game-theoretic probability

A recurring theme in Glenn's research is the idea of going back to betting as a source of intuition for probability theory and statistics [15,22,19,21]. The static picture of betting was very prominent, as he often emphasizes (see, e.g., [15, Section 2.1] and his first paper in this Special Issue), already in the work by Bruno de Finetti and his subjectivist colleagues. Glenn's

* Corresponding author.
E-mail addresses: john.aldrich@soton.ac.uk (J. Aldrich), apd@statslab.cam.ac.uk (A. Philip Dawid), thierry.denoeux@utc.fr (T. Denoeux), pshenoy@ku.edu (P.P. Shenoy), v.vovk@rhul.ac.uk (V. Vovk).

https://doi.org/10.1016/j.ijar.2022.01.003

work on the foundations of conditional probability [12,13] clearly showed the necessity of having a well-defined protocol, a probability tree, for learning new information in order for conditional probabilities to have a clear interpretation. As a result, he created a dynamic picture of betting in probability trees [13,16]. Interestingly, it all started from his interest in the history of probability, namely in Thomas Bayes's thought [12].

The dynamic story of betting is very closely connected with Jean Ville's [24] notion of a martingale, namely with *test martingales*, i.e., nonnegative martingales with initial value 1. Jean Ville and his research have become one of the favorite topics of Glenn's research (see Section 5 of his biography in this Special Issue).

Motivated by Dawid's prequential principle [3,4], Shafer and Vovk [22] made another step in incorporating betting into probability theory and statistics, proposing to start from nonnegative martingales as the basis and to define probabilities in their terms. Such *game-theoretic probabilities* are usually imprecise, especially in discrete time, but in several common cases upper and lower probabilities are not very different.

In the recent paper [21], Glenn developed the dynamic picture of betting as an alternative to the usual language of significance tests. In this context, the factor by which the bettor has multiplied the capital risked is an alternative to the p-value as a measure of evidence. Glenn's term for this factor was "betting score", but the term "e-value" introduced in [26] is also used, including by two papers in this Special Issue.

An exciting direction of research in game-theoretic probability is continuous-time processes, which is the topic of Part IV of [23]. The usual way in which the theory of stochastic processes finds its way into applications is via stochastic differential equations, treated in Rafał M. Łochowski, Nicolas Perkowski, and David J. Prömel's "Game-theoretic integration and differential equation" [9]. The authors prove, among other results, the game-theoretic counterpart of the famous Yamada-Watanabe theorem about the existence and uniqueness of solutions to stochastic differential equations under weak conditions on the coefficients. Being game-theoretic, this is a worst-case result about continuous-time processes, and thus it does not assume their stochasticity in the usual sense, mathematical or philosophical.

One of the origins of Shafer and Vovk's joint work [22,25,23] was the algorithmic theory of randomness, originated by Kolmogorov in the early 1960s. Kolmogorov's work was extended in the direction of betting by Schnorr and Levin, but those authors worked in the context of precise probabilities. Gert de Cooman and Jasper De Bock in "Randomness is inherently imprecise" [5] explore the notion of randomness for imprecise probabilities. This is an important extension for which the authors prove a plethora of new results, and they show that it is far from vacuous: there are sequences of observations that are random with respect to an imprecise probability forecasting system but are not random with respect to any probability forecasting system with a smaller degree of imprecision, in a natural sense (let alone with respect to a precise probability forecasting system).

In "Log-optimal anytime-valid e-values" [7], Wouter Koolen and Peter Grünwald explore the efficiency of anytime-valid e-values, starting from a fundamental result on primal-dual equivalence. This result implies, for example, closed-form expressions for the case of a simple null hypothesis and the case of the Gaussian distributions with mean zero and arbitrary variance.

The last paper in this part is "Testing exchangeability: Fork-convexity, supermartingales and e-processes" by Aaditya Ramdas, Johannes Ruf, Martin Larsson, and Wouter Koolen [10]. In this paper, the authors develop powerful ways of testing the assumption of exchangeability by dynamic betting in the case of binary observations. The proposed testing procedure is not exactly a test martingale, but is close and involves a generalization, which they call a safe e-process. The authors not only state the efficiency result for their testing procedure in the standard measure-theoretic terms, but also state it in terms of game-theoretic probability. This paper can be thought of as a gentle transition to the following part of the Special Issue, conformal prediction. Testing exchangeability is a big topic in conformal prediction [25, Section 7.1], and, as the authors point out, conformal test martingales can also be applied to answering their question (with a somewhat weaker notion of safety, but extending in a straightforward way to complex observations).

Conformal prediction

Together with his colleagues, Glenn developed a new area of machine learning, conformal prediction, in the early 2000s and published its book-length exposition as [25]. This work is very different from Glenn's work in game-theoretic probability in that conformal prediction only works under fairly restrictive assumptions, usually under the assumption of exchangeability. This assumption, however, is standard in machine learning and much of nonparametric statistics.

An appealing property of conformal prediction is its guaranteed validity under the assumption of exchangeability. In their paper "Validity, consonant plausibility measures, and conformal prediction" [2], Leonardo Cella and Ryan Martin offer a new analysis of achievable notions of validity and show that conformal prediction satisfies a new property of validity.

The title of Minge Xie and Zheshi Zheng's "Homeostasis phenomenon in conformal prediction and predictive distribution function" [27] draws an interesting analogy between conformal prediction and biology. While homeostasis in living systems means maintaining key variables, such as temperature, in a preset range, homeostasis in conformal prediction refers to maintaining predictive validity, such as a preset probability of error for prediction intervals, regardless of the validity of the learning model used as input for conformal prediction. In their picture of conformal prediction, the bias introduced by a wrong model is offset automatically to maintain validity, which is the key of the homeostasis phenomenon. Their study explores regions where the homeostasis property breaks down.

History of probability and statistics

Glenn's interest in the foundations of probability and statistics has been complemented and reinforced by a concern with history. Practitioners are often occasional historians, available for commemorations and ready to note the ancestry of their own contributions, but attention to history is an essential part of Glenn's scientific personality. He develops an idea and asks deep historical questions about it with the history informing further developments. The pattern was there from the beginning in 1976–1978 with the historical "Non-additive probabilities in the work of Bernoulli and Lambert" complementing his *Mathematical Theory of Evidence*. It continues with his interest in Cournot and Ville.

The historical part of the Special Issue contains two papers. Both are about individuals and their place in "mainstream" probability. John Aldrich's "W. E. Johnson and Cambridge thought on probability" [1] is devoted to William Ernest Johnson, a figure at once familiar and obscure and perhaps best remembered for his early use of exchangeability. Early Japanese work on probability is an unfamiliar field for Western scholars and is explored in Clémentine Laurens and Laurent Mazliak's "Kameda Toyojiro and the transfer of the Western theory of probability to Japan" [8]. Kameda played a major role in initiating the impressive development of probability and statistics in Japan after it entered the Meiji era. Glenn has been interested in Japan for a long time, and his ideas of betting as a possible foundation for probability found a sympathetic reception among Japanese statisticians, first of all Kei Takeuchi and Akimichi Takemura and their students, and he has visited Japan several times to attend workshops on game-theoretic probability organized by them.

Glenn's papers published in this Special Issue

This Special Issue presents for a wider audience three papers that were originally prepared by Glenn in 2001–2018. The first of these papers, "The notion of event in probability and causality: Situating myself relative to Bruno de Finetti", was prepared for Glenn's talks in Pisa and Bologna in March 2001. On one hand, Glenn acknowledges his substantial intellectual debt to de Finetti: "De Finetti's most fundamental and enduring insight, it seems to me, is the insight that probability, in all its ramifications and applications, is inextricably tied to the idea of betting." Where he disagrees with de Finetti is that Glenn prefers the dynamic picture of betting ("I venture to say that dynamics is central to the intuitive picture of probability theory"), which he started in 1982 in his analysis of protocols implicit in Bayes's work. This analysis leads to a sharp distinction between two different roles of events, which can be bearers of probabilities and givers of probabilities. The two roles are performed by different kinds of events.

The second paper, "Causal interpretation of graphical models", was prepared for his invited talk at the 54th Session of the International Statistical Institute held in August 2003 in Berlin. Causality and its formalizations in terms of graphical models had been among Glenn's main research interests: see, e.g., his books [17] and [18], which can be consulted for further information. In this paper he explains that the picture of game-theoretic probability drawn in his 2001 book joint with Vovk [22] sheds new light on the meaning of causality in the context of graphical models. The paper is reprinted (in a modified form) with a kind permission from the International Statistical Institute.

Glenn's final paper in this Special Issue, "'So much data. Who needs probability?' Have we been here before?", is based on an after-dinner talk given at the Fifth Bayesian, Fiducial, and Frequentist (BFF) Conference in 2018. The BFF community is trying to explore connections between and unify various approaches to statistics. Probability is fundamental to all three approaches listed in its name, and the question that Glenn discusses in the talk is "what made us think that statistics needs probability"? His answer to the version of the question with "what" substituted by "who" is "Antoine Augustin Cournot (1801–1877)".

References

[1] John Aldrich, W.E. Johnson and Cambridge thought on probability, Int. J. Approx. Reason. (2021), https://doi.org/10.1016/j.ijar.2021.10.010.
[2] Leonardo Cella, Ryan Martin, Validity, consonant plausibility measures, and conformal prediction, Int. J. Approx. Reason. (2021), https://doi.org/10.1016/j.ijar.2021.07.013.
[3] A. Philip Dawid, Statistical theory: the prequential approach (with discussion), J. R. Stat. Soc. A 147 (1984) 278–292.
[4] A. Philip Dawid, Calibration-based empirical probability (with discussion), Ann. Stat. 13 (1985) 1251–1285.
[5] Gert de Cooman, Jasper De Bock, Randomness is inherently imprecise, Int. J. Approx. Reason. (2021), https://doi.org/10.1016/j.ijar.2021.06.018.
[6] Thierry Denoeux, 40 years of Dempster-Shafer theory, Int. J. Approx. Reason. 79 (2016) 1–6.
[7] Wouter M. Koolen, Peter Grünwald, Log-optimal anytime-valid e-values, Int. J. Approx. Reason. (2021), https://doi.org/10.1016/j.ijar.2021.09.010.
[8] Clémentine Laurens, Laurent Mazliak, Kameda Toyojiro and the transfer of the Western theory of probability to Japan, Int. J. Approx. Reason. (2021), https://doi.org/10.1016/j.ijar.2021.10.006.
[9] Rafał M. Łochowski, Nicolas Perkowski, David J. Prömel, Game-theoretic integration and differential equation, Int. J. Approx. Reason. (2021).
[10] Aaditya Ramdas, Johannes Ruf, Martin Larsson, Wouter M. Koolen, Testing exchangeability: fork-convexity, supermartingales and e-processes, Int. J. Approx. Reason. (2021), https://doi.org/10.1016/j.ijar.2021.06.017.
[11] Glenn Shafer, A Mathematical Theory of Evidence, Princeton University Press, Princeton, NJ, 1976.
[12] Glenn Shafer, Bayes's two arguments for the rule of conditioning, Ann. Stat. 10 (1982) 1075–1089.
[13] Glenn Shafer, Conditional probability (with discussion), Int. Stat. Rev. 53 (1985) 261–277.
[14] Glenn Shafer, Perspectives on the theory and practice of belief functions, Int. J. Approx. Reason. 4 (1990) 323–362.
[15] Glenn Shafer, What is probability?, in: David C. Hoaglin, David S. Moore (Eds.), Perspectives on Contemporary Statistics, vol. 21, Mathematical Association of America, 1992, pp. 93–105. Also published as Working Paper 230, School of Business, University of Kansas, Lawrence, KS, August 1991.
[16] Glenn Shafer, Can the various meanings of probability be reconciled?, in: Gideon Keren, Charles Lewis (Eds.), A Handbook for Data Analysis in the Behavioral Sciences: Methodological Issues, Lawrence Erlbaum, Hillsdale, NJ, 1993, pp. 165–196. Also published as Working Paper 230, School of Business, University of Kansas, Lawrence, KS, August 1991.

[17] Glenn Shafer, The Art of Causal Conjecture, MIT Press, Cambridge, MA, 1996.
[18] Glenn Shafer, Probabilistic Expert Systems, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 67, SIAM, Philadelphia, 1996.
[19] Glenn Shafer, A betting interpretation for probabilities and Dempster-Shafer degrees of belief, Int. J. Approx. Reason. 52 (2011) 127–136.
[20] Glenn Shafer, A Mathematical Theory of Evidence turns 40, Int. J. Approx. Reason. 79 (2016) 7–25.
[21] Glenn Shafer, The language of betting as a strategy for statistical and scientific communication (with discussion), J. R. Stat. Soc. A 184 (2021) 407–478.
[22] Glenn Shafer, Vladimir Vovk, Probability and Finance: It's Only a Game!, Wiley, New York, 2001.
[23] Glenn Shafer, Vladimir Vovk, Game-Theoretic Foundations for Probability and Finance, Wiley, Hoboken, NJ, 2019.
[24] Jean Ville, Etude critique de la notion de collectif, Gauthier-Villars, Paris, 1939.
[25] Vladimir Vovk, Alex Gammerman, Glenn Shafer, Algorithmic Learning in a Random World, Springer, New York, 2005.
[26] Vladimir Vovk, Ruodu Wang, E-values: calibration, combination, and applications, Ann. Stat. 49 (2021) 1736–1754.
[27] Min-ge Xie, Zheshi Zheng, Homeostasis phenomenon in conformal prediction and predictive distribution functions, Int. J. Approx. Reason. (2021), https://doi.org/10.1016/j.ijar.2021.09.001.