A Note on Factorization of Belief Functions

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Abstract. Practically all methods for efficient computation with multidimensional models take advantage of the fact that the model in question in a way factorizes. It means that it is possible to decompose the model into its low-dimensional parts, each of which can be defined with a *reasonable* number of parameters. This is a basic idea of computation with probabilistic Graphical Markov Models as well as with belief or credal networks. For belief functions, two types of factorization were designed in the literature: one is based on the famous Dempster's rule of combination, the other uses an operator of composition. The paper compares these two types of factorization, shows that both the approaches are equivalent to each other in case of unconditional factorization and shows what are the differences when overlapping factorization is studied.

Keywords: belief functions, conditional independence, irrelevance, multidimensional models

1 Probabilistic motivation

In probability theory, notions of independence, factorization, irrelevance and conditional probability are closely interconnected [4, 10]. Consider two random variables X and Y and their joint probability distribution $\pi(X,Y)$. We say X and Y are *independent* with respect to π, written as $X \perp\!\!\!\perp Y\ [\pi]$, *iff*

$$\pi(X,Y) = \pi(X) \cdot \pi(Y), \tag{1}$$

where $\pi(X) = \pi^{\downarrow X}$, $\pi(Y) = \pi^{\downarrow Y}$ denote the respective marginal one-dimensional distributions. It means that the joint distribution $\pi(X,Y)$ is uniquely represented just by its one-dimensional marginals. We also say that $\pi(X,Y)$ *factorizes* into its respective marginals. It is well-known that if the conditional distribution $\pi(X|Y)$ is defined then $X \perp\!\!\!\perp Y\ [\pi]$ *iff*

$$\pi(X|Y) = \pi(X). \tag{2}$$

The last formula expresses the fact that knowledge of a value of variable Y does not change our knowledge regarding the value of variable X. Therefore, we will refer to this property as *irrelevance*. In connection with this two-dimensional case we want to recall one more property, usually called *factorization lemma*, saying that $X \perp\!\!\!\perp Y\ [\pi]$ *iff* there exist two real-valued functions ϕ and ψ defined on the sets of values of variables X and Y, respectively, such that for all tuples $(x,y) \in \mathbb{X} \times \mathbb{Y}$ ($\mathbb{X}, \mathbb{Y}$ denote the sets of values of variables X and Y, respectively)

$$\pi(X=x, Y=y) = \phi(x) \cdot \psi(y). \tag{3}$$

All these concepts generalize also to the case of conditional independence. Consider a joint distribution $\pi(X,Y,Z)$ of three random variables. We say that X and Y are conditionally independent given Z, written as $X \perp\!\!\!\perp Y|Z\ [\pi]$, *iff*

$$\pi(X,Y,Z) \cdot \pi(Z) = \pi(X,Z) \cdot \pi(Y,Z), \tag{4}$$

which is known to be equivalent to (in case that $\pi(Y|Z)$ is defined)

$$\pi(X,Y,Z) = \pi(X,Z) \cdot \pi(Y|Z). \tag{5}$$

It means that $X \perp\!\!\!\perp Y|Z\ [\pi]$ *iff* $\pi(X,Y,Z)$ factorizes into its (conditional) marginals. Nevertheless, due to the famous factorization lemma it is known that it is also equivalent to the fact that there exist two real-valued functions ϕ and ψ defined on the sets $\mathbb{X} \times \mathbb{Z}$ and $\mathbb{Y} \times \mathbb{Z}$, respectively, such that for all triples $(x,y,z) \in \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$

$$\pi(X=x, Y=y, Z=z) = \phi(x,z) \cdot \psi(y,z). \tag{6}$$

Thus, in probability theory, conditional independence and both types of factorization (i.e., factorization into marginals and general factorization into "potential functions") are fully equivalent. And, let us stress that if conditional probability distributions $\pi(X|Y)$ and $\pi(X|Y,Z)$ are defined, it is also equivalent to the notion of irrelevance, which correspond in this conditional case to the formula

$$\pi(X|Y,Z) = \pi(X|Z). \tag{7}$$

So, from probability theory we are accustomed to the fact that the notions of factorization (both types), conditional independence and conditional irrelevance fully coincide. The goal of this paper is to show on a couple of examples that for belief functions the situation is much more complicated.

2 Basic notions and notation

2.1 Set notation

As in the previous section we will consider three variables X, Y, Z having finite sets of values $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$, respectively.

A *projection* of $a = (x, y, z) \in \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$ into $\mathbb{X}$ will be denoted $a^{\downarrow X}$, i.e., $a^{\downarrow X} = x$. Analogously, $a^{\downarrow XY} = (x, y)$. Similarly, for $A \subset \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$, $A^{\downarrow Z}$ will denote a *projection* of A into $\mathbb{Z}$:

$$A^{\downarrow Z} = \{z \in \mathbb{Z} \mid \exists a \in A : z = a^{\downarrow Z}\}.$$

Analogously,

$$A^{\downarrow XZ} = \{(x, z) \in \mathbb{X} \times \mathbb{Z} \mid \exists a \in A : (x, z) = a^{\downarrow XZ}\}.$$

Let us remark that $A^{\downarrow \emptyset} = \emptyset$.

In addition to the projection, in this text we will need also an opposite operation, which will be called a joint. By a *joint* of two sets $A \subseteq \mathbb{X} \times \mathbb{Z}$ and $B \subseteq \mathbb{Y} \times \mathbb{Z}$ we will understand a set

$$A \bowtie B = \{a \in \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} : a^{\downarrow XZ} \in A \;\; \& \;\; a^{\downarrow YZ} \in B\}.$$

Let us add that for $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$ we get

$$A \bowtie B = A \times B,$$

and for $A, B \subset \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$

$$A \bowtie B = A \cap B.$$

2.2 Basic notation for belief functions

In evidence theory (or Dempster-Shafer theory) two measures are used to model the uncertainty: belief and plausibility measures. Both of them can be defined with the help of another set function called a *basic (probability* or *belief) assignment* m. It is a set function

$$m : \mathcal{P}(\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}) \longrightarrow [0, 1],$$

($\mathcal{P}(\mathbb{X} \times \mathbb{Y} \times \mathbb{Z})$ denotes a power set of $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$), for which $\sum_{A \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}} m(A) = 1$. Furthermore, we assume that $m(\emptyset) = 0$. A set $A \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$ is said to be a *focal element* of m if $m(A) > 0$.

In addition to *belief* and *plausibility* measures, which will not be discussed in this paper, also *commonality function* can be obtained from basic assignment m:

$$Q(A) = \sum_{B \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} : A \subseteq B} m(B).$$

For a basic assignment m on $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$, its *marginal basic assignments* will be used. For example (for each $A \subseteq \mathbb{X}$),

$$m^{\downarrow X}(A) = \sum_{B \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Z} : B^{\downarrow X} = A} m(B).$$

Analogously, $Q^{\downarrow X}$ will denote the respective marginal commonality function.

Regarding the Dempster-Shafer theory of evidence, perhaps the most important notion is the Dempster's rule of combination ([5]) defined for two basic assignments m_1 and m_2 on $\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$ by the formula (for all $A\subseteq\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$)

$$m_1\oplus m_2(A)=\frac{1}{K}\sum_{B,C\subseteq\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}:A=B\cap C} m_1(B)\cdot m_2(C),$$

where $K=1-\sum_{B,C\subseteq\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}:B\cap C=\emptyset} m_1(B)\cdot m_2(C)$. For the purpose of this paper it is important to stress that this rule of combination is defined also for basic assignments which are not defined on the same space of discernment. For example, for m_1 defined on $\mathbb{X}\times\mathbb{Z}$ and m_2 defined on $\mathbb{Y}\times\mathbb{Z}$ their combination is defined for all $A\subseteq\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$

$$m_1\oplus m_2(A)=\frac{1}{K}\sum_{B\subseteq\mathbb{X}\times\mathbb{Z},C\subseteq\mathbb{Y}\times\mathbb{Z}:A=B\bowtie C} m_1(B)\cdot m_2(C), \tag{8}$$

where $K=1-\sum_{B\subseteq\mathbb{X}\times\mathbb{Z},C\subseteq\mathbb{Y}\times\mathbb{Z}:B\bowtie C=\emptyset} m_1(B)\cdot m_2(C)$. Similarly, for m_1 defined on $\mathbb{X}\times\mathbb{Y}$ and m_2 defined on $\mathbb{Y}$ their combination is defined for all $A\subseteq\mathbb{X}\times\mathbb{Y}$

$$m_1\oplus m_2(A)=\frac{1}{K}\sum_{B\subseteq\mathbb{X}\times\mathbb{Y},C\subseteq\mathbb{Y}:A=B\bowtie C} m_1(B)\cdot m_2(C), \tag{9}$$

where, again, $K=1-\sum_{B\subseteq\mathbb{X}\times\mathbb{Y},C\subseteq\mathbb{Y}:B\bowtie C=\emptyset} m_1(B)\cdot m_2(C)$.

It is well-known [8] that the Dempster's rule of combination, when observed through commonality functions, manifests itself as a simple multiplication of the respective commonality functions. For example, when considering m_1 and m_2 from formula (9), this expression simplifies to

$$(Q_1\oplus Q_2)(A)=Q_1(A)\cdot Q_2(A^{\downarrow Y}).$$

This enables us to define an operation that is an inverse of the Dempster's rule of combination. Consider a basic assignment m defined on $\mathbb{X}\times\mathbb{Y}$. By $m\ominus m^{\downarrow Y}$ we understand that basic assignment whose commonality function is defined by the following expression

$$(Q\ominus Q^{\downarrow Y})(A)=Q(A)/Q^{\downarrow Y}(A^{\downarrow Y}). \tag{10}$$

Such an inverse operation enables us to remove the information contained in $m^{\downarrow Y}$ from the joint basic assignment m. This is why we can call this basic assignment *conditional basic assignment* and will denote it sometimes $m(X|Y)$. However, it is important to realize that $Q\ominus Q^{\downarrow Y}$ is not always a basic assignment.

2.3 Example 1

Let X,Y be binary variables with values in $\mathbb{X}=\{\xi,\bar{\xi}\}$, $\mathbb{Y}=\{\eta,\bar{\eta}\}$. Consider first basic assignment m_1 with two focal elements: $m_1(\{(\xi,\eta)\})=0.4$ and $m_1(\{(\xi,\eta),(\xi,\bar{\eta})\})=0.6$. The reader can easily verify that its marginal $m_1^{\downarrow Y}$ has two focal elements: $m_1^{\downarrow Y}(\{\eta\})=0.4$ and $m_1^{\downarrow Y}(\{\eta,\bar{\eta}\})=0.6$, and that the respective commonality function is $Q_1^{\downarrow Y}(\{\eta\})=1, Q_1^{\downarrow Y}(\{\bar{\eta}\})=0.6, Q_1^{\downarrow Y}(\{\eta,\bar{\eta}\})=0.6$. The commonality function for m_1 is in Table 2.3 (there is no focal element of m_1 with more than two couples from $\mathbb{X}\times\mathbb{Y}$, which means that for all three or four-point sets, which are not explicitly described in Table 2.3, the respective commonality function equals 0). Thus, using formula (10) we can easily compute the commonality function corresponding to $m_1\ominus m_1^{\downarrow Y}$, which is a conditional basic assignment $m(X|Y)$. All these computations are depicted in Table 2.3.

Considering another basic assignment m_2 on the same frame of discernment $\mathbb{X}\times\mathbb{Y}$ having again only two focal elements: $m_2(\{(\xi,\bar{\eta}),(\bar{\xi},\bar{\eta})\})=m_2(\{(\xi,\bar{\eta}),(\bar{\xi},\eta)\})=0.5$, one can immediately see that $m_2^{\downarrow Y}$ has again two focal elements, namely $m_2(\{\bar{\eta}\})=m_2(\{\eta,\bar{\eta}\})=0.5$. To compute $Q_2\ominus Q_2^{\downarrow Y}$ we need commonality function $Q_2^{\downarrow Y}$ which is (because $m_2^{\downarrow Y}(\{\eta\})=0$): $Q_2^{\downarrow Y}(\{\eta\})=0.5, Q_2^{\downarrow Y}(\{\bar{\eta}\})=1, Q_2^{\downarrow Y}(\{\eta,\bar{\eta}\})=0.5$. But this time trying to get $m_2(X|Y)=m_2\ominus m_2^{\downarrow Y}$ one gets the only solution for which $m_2(X=\xi|Y=\bar{\eta})=-0.5$, which does not comply with the definition of a basic assignment.

Table 1. Basic assignments and their commonality functions

A	m_1	Q_1	$Q_1 \ominus Q_1^{\downarrow Y}$	$m_1(X\|Y)$	m_2	Q_2	$Q_2 \ominus Q_2^{\downarrow Y}$	$m_2(X\|Y)$
(ξ,η)	0.4	1	1	0	0	0	0	0
$(\xi,\bar\eta)$	0	0.6	1	0	0	1	$\frac{1}{1}=1$	-0.5
$(\bar\xi,\eta)$	0	0	0	0	0	0.5	$\frac{0.5}{0.5}=1$	0
$(\bar\xi,\bar\eta)$	0	0	0	0	0	0.5	$\frac{0.5}{1}=0.5$	0
$(\xi,\eta),(\xi,\bar\eta)$	0.6	0.6	1	1	0	0	0	0
$(\xi,\eta),(\bar\xi,\eta)$	0	0	0	0	0	0	0	0
$(\xi,\eta),(\bar\xi,\bar\eta)$	0	0	0	0	0	0	0	0
$(\xi,\bar\eta),(\bar\xi,\eta)$	0	0	0	0	0.5	0.5	$\frac{0.5}{0.5}=1$	1
$(\xi,\bar\eta),(\bar\xi,\bar\eta)$	0	0	0	0	0.5	0.5	$\frac{0.5}{1}=0.5$	0.5
$(\bar\xi,\eta),(\bar\xi,\bar\eta)$	0	0	0	0	0	0	0	0

3 Unconditional independence and factorization

After reading the section on probabilistic motivation, it is perhaps not so surprising that a generally accepted notion of (unconditional, or marginal) independence can be introduced in several ways [1, 3, 9, 10]. Perhaps the simplest one is the following one.

Definition 1. Let m be a basic assignment on $\mathbb{X}\times\mathbb{Y}$. We say that variables X and Y are *independent*, written as $X \perp\!\!\!\perp Y\ [m]$, if for all $A \subseteq \mathbb{X}\times\mathbb{Y}$

$$Q(A) = Q^{\downarrow X}(A^{\downarrow X}) \cdot Q^{\downarrow Y}(A^{\downarrow Y}).$$

In [7], the following assertion was proved.

Proposition 1. $X \perp\!\!\!\perp Y\ [m]$ iff

$$m(A) = m^{\downarrow X}(A^{\downarrow X}) \cdot m^{\downarrow Y}(A^{\downarrow Y}) \tag{11}$$

for all $A \subseteq \mathbb{X}\times\mathbb{Y}$ *for which* $A = A^{\downarrow X}\times A^{\downarrow Y}$*, and* $m(A)=0$ *otherwise.*

Thus we see that for independent variables both the basic assignment and the commonality function factorize like a probability distribution factorizes for independent random variables. Nevertheless, since we are studying notions of independence within the Dempster-Shafer theory of evidence, the following assertion is of great importance.

Proposition 2. $X \perp\!\!\!\perp Y\ [m]$ iff

$$m = m^{\downarrow X} \oplus m^{\downarrow Y}. \tag{12}$$

The question remains whether there is some property that would be a belief function counterpart to probabilistic irrelevance expressed by formula (2). And it is an easy task to show that such a property holds true. Because

$$m \ominus m^{\downarrow Y} = m^{\downarrow X} \oplus m^{\downarrow Y} \ominus m^{\downarrow Y} = m^{\downarrow X}. \tag{13}$$

The above modifications are based on associativity of Dempster's rule of combination and its inverse, which follows trivially from the associativity of multiplication and division of commonality functions.

Remark 1. From equalities (13) one can see that if $X \perp\!\!\!\perp Y\ [m]$ than $m \ominus m^{\downarrow Y}$ is always defined. Nevertheless, the reader certainly noticed that while $m \ominus m^{\downarrow Y}$ is defined on $\mathbb{X}\times\mathbb{Y}$, the marginal $m^{\downarrow X}$ is defined only on $\mathbb{X}$. This discrepancy can be easily explained: in this case $m \ominus m^{\downarrow Y}$ is always a "vacuous extension" of $m^{\downarrow X}$, which means that if A is a focal element of $m^{\downarrow X}$, then $A\times\mathbb{Y}$ is a focal element of $m \ominus m^{\downarrow Y}$, and $m^{\downarrow X}(A) = m \ominus m^{\downarrow Y}(A\times\mathbb{Y})$.

Though it will not be shown until in the next paragraph (Proposition 5), let us mention already here that for this unconditional case even the general factorization lemma holds true (its probabilistic version is expressed by formula (3)). So we can summarize that unconditional independence and factorization for belief functions (or rather for basic assignments) manifests the same properties as within the framework of probability theory (including the fact that a conditional basic assignment is not always unambiguously defined).

4 Conditional independence and factorization

Let us start this section with presenting the definition which is used by many authors (though different authors call the notion in different ways) [2, 3, 9] and which is different from the one introduced in [7].

Definition 2. Let m be a basic assignment on $\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$. We say that variables X and Y are *conditionally independent given* Z (in notation $X \perp\!\!\!\perp Y|Z\ [m]$) if for all $A \subseteq \mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$

$$Q(A)\cdot Q^{\downarrow Z} = Q^{\downarrow XZ}(A^{\downarrow XZ})\cdot Q^{\downarrow YZ}(A^{\downarrow YZ}).$$

Taking into account properties of commonality functions, one gets almost immediately the following property that forms a conditional version of Proposition 2.

Proposition 3. $X \perp\!\!\!\perp Y|Z\ [m]$ iff

$$m = m^{\downarrow XZ} \oplus m^{\downarrow YZ} \ominus m^{\downarrow Z}, \tag{14}$$

if the right hand part of this formula is defined.

Let us present one more property that was proved in [2] and that will appear quite important from the point of view of factorization. We will call it a *weak factorization lemma*.

Proposition 4. *If* $X \perp\!\!\!\perp Y|Z\ [m]$ *than all focal elements* A *of* m *are of the form* $A = A^{\downarrow XZ} \bowtie A^{\downarrow YZ}$.

4.1 General factorization

Let us start studying the question whether there is a relation between $X \perp\!\!\!\perp Y|Z\ [m]$ and an existence of two real-valued set functions

$$\phi : \mathcal{P}(\mathbb{X}\times\mathbb{Z}) \longrightarrow [0,+\infty),$$
$$\psi : \mathcal{P}(\mathbb{Y}\times\mathbb{Z}) \longrightarrow [0,+\infty),$$

such that for all $A \subseteq \mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$

$$m(A) = \phi(A^{\downarrow XZ})\cdot\psi(A^{\downarrow YZ}). \tag{15}$$

First of all, realize that if we want to factorize a basic assignment m for which the conditional independence $X \perp\!\!\!\perp Y|Z\ [m]$ should hold, then, due to the above presented weak independence lemma (Proposition 4), we must require validity of equality (15) only for the sets that can be expressed as a join of their projections: $A = A^{\downarrow XZ} \bowtie A^{\downarrow YZ}$. In what follows we will take advantage of the following factorization lemma that was first proved in [11]. However, in this assertion a new type of a binary operator appears, which is called an operator of composition, and therefore its definition has to precede the presentation of the mentioned factorization lemma.

Definition 3 (Operator of composition). Let m_1 be a basic assignment on $\mathbb{X}\times\mathbb{Z}$ and m_2 on $\mathbb{Y}\times\mathbb{Z}$. A *composition* $m_1 \triangleright m_2$ is defined for each $C \subseteq \mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$ by one of the following expressions:

[a] if $m_2^{\downarrow Z}(C^{\downarrow Z}) > 0$ and $C = C^{\downarrow XZ} \bowtie C^{\downarrow YZ}$ then

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow XY})\cdot m_2(C^{\downarrow YZ})}{m_2^{\downarrow Z}(C^{\downarrow Z})};$$

[b] if $m_2^{\downarrow Z}(C^{\downarrow Z}) = 0$ and $C = C^{\downarrow XZ} \times \mathbb{Y}$ then

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow XZ});$$

[c] in all other cases $(m_1 \triangleright m_2)(C) = 0$.

Proposition 5 (General factorization lemma). *Let* m *be a basic assignment on* $\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$. *Then there exist real-valued set functions*

$$\phi : \mathcal{P}(\mathbb{X}\times\mathbb{Z}) \longrightarrow [0,+\infty),$$
$$\psi : \mathcal{P}(\mathbb{Y}\times\mathbb{Z}) \longrightarrow [0,+\infty),$$

such that for all $A \subseteq \mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$

$$m(A) = \begin{cases} \phi(A^{\downarrow XZ})\cdot\psi(A^{\downarrow YZ}) & \text{if } A = A^{\downarrow XZ} \bowtie A^{\downarrow YZ}, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

iff $m = m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$.

Remark 2. Notice that for basic assignments m_1 on $\mathbb{X}$ and m_2 on $\mathbb{Y}$ the definition of $m_1 \triangleright m_2$ simplifies to

$$m_1 \triangleright m_2(A) = \begin{cases} m_1(A^{\downarrow X}) \cdot m_2(A^{\downarrow Y}) & \text{if } A = A^{\downarrow X} \times A^{\downarrow Y}, \\ 0 & \text{otherwise,} \end{cases}$$

which corresponds to the definition of $X \perp\!\!\!\perp Y \; [m_1 \triangleright m_2]$. Therefore, for unconditional independence, Proposition 5 is a characterization of the general factorization promised at the end of Section 3.

4.2 Comparison of the Dempster's rule of combination and the operator of composition

First of all, let us stress that the operator of composition is something other than the Dempster's rule of combination (or its non-normalized version, the so called *conjunctive combination rule* [1]). For example, the operation of composition is neither commutative nor associative. While Dempster's rule of combination was designed to combine different (independent) sources of information (it realizes fusion of sources), the operator of composition primarily serves for composing pieces of local information (usually coming from one source) into a global model. The difference between these two tasks can be illustrated by the following analogy. The Dempster's rule of combination is appropriate when one wants to reconstruct an image of a person from a number of fuzzy, out of focus pictures. On the other hand, the operator of composition is to be used when one wants to reassemble a picture that was teared into a great number of small pieces.

From what was said it is perhaps intuitively obvious that the notion of composition is in a way naturally connected with the notion of factorization. This fact manifests also in the following difference: while for computation of $m_1 \triangleright m_2(C)$ it is enough to know only m_1 and m_2 just for the respective projections of set C, computing $m_1 \oplus m_2(C)$ requires knowledge of, roughly speaking, the entire basic assignments m_1 and m_2.

For further intuitive justification of the operator of composition the reader is referred to [6, 7], where a number of its properties were proved. Let us present just a couple of the most important ones.

Proposition 6. ***Basic Properties.*** *Let m_1 and m_2 be basic assignments defined on $\mathbb{X} \times \mathbb{Z}$ and $\mathbb{Y} \times \mathbb{Z}$, respectively. Then:*

1. *$m_1 \triangleright m_2$ is a basic assignment on $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$;*
2. *$(m_1 \triangleright m_2)^{\downarrow XZ} = m_1$;*
3. *$m_1 \triangleright m_2 = m_2 \triangleright m_1 \iff m_1^{\downarrow Z} = m_2^{\downarrow Z}$.*

The reader probably noticed that property 2 guarantees idempotency of the operator and gives a hint about how to get a counterexample to its commutativity. From point 1, one immediately gets an idea in what way it enables us to construct multidimensional basic assignments by an iterative application of the operator of composition.

As it will be shown in the following assertions, in spite of all these differences, in special (rather degenerated) situations the two discussed combination rules coincide.

Proposition 7. *Let m_1 be a basic assignment on $\mathbb{X}$ and m_2 be an assignment on $\mathbb{Y}$. Then*

$$m_1 \oplus m_2 = m_1 \triangleright m_2.$$

Proof. The proof is almost obvious. For $A = A^{\downarrow X} \bowtie A^{\downarrow Y}$ the Dempster's rule of combination yields

$$m_1 \oplus m_2(A) = \frac{1}{K} \sum_{B \subseteq \mathbb{X}, C \subseteq \mathbb{Y} : A = B \times C} m_1(B) \cdot m_2(C) = m_1(A^{\downarrow X}) \cdot m_2(A^{\downarrow Y}),$$

(notice that in this case $K = 1$) and the same expression we get directly from the definition of the operator of composition

$$m_1 \triangleright m_2(A) = \frac{m_1(A^{\downarrow X}) \cdot m_2(A^{\downarrow Y})}{m_2(\emptyset)} = m_1(A^{\downarrow X}) \cdot m_2(A^{\downarrow Y}).$$

For $A \neq A^{\downarrow X} \bowtie A^{\downarrow Y}$, both $m_1 \oplus m_2(A)$ and $m_1 \triangleright m_2(A)$ equal 0: for the Dempster's rule of combination it follows from Proposition 4, for the operator of composition it follows directly from the definition. $\square$

It is well-known that Dempster-Shafer theory of evidence is an extension of both probability and possibility theories [8]. Possibilistic measures are represented by basic assignments whose all focal elements are *nested* (they can be ordered so that the preceding one is a subset of the subsequent focal element). Probability measures are represented by basic assignments whose all focal elements are singletons (one-point-sets). These basic assignments are usually called Bayesian. One can easily see that for Bayesian basic assignments $m(A) = Q(A)$ for all sets A, and that all marginals of a Bayesian basic assignment are also Bayesian.

Proposition 8. *Consider two Bayesian basic assignments m_1 and m_2 defined on $\mathbb{X} \times \mathbb{Z}$ and $\mathbb{Y} \times \mathbb{Z}$, respectively. If $m_2^{\downarrow Z}(z) = 0 \implies m_1^{\downarrow Z}(z) = 0$ then*

$$m_1 \oplus m_2 \ominus m_2^{\downarrow Z} = m_1 \triangleright m_2.$$

Proof. Consider a point $a \in \mathbb{X}\times\mathbb{Y}\times\mathbb{Z}$. Since we assume that m_1 is Bayesian then $m_1(a^{\downarrow XZ}) = Q_1(a^{\downarrow XZ})$, and, analogously, $m_2(a^{\downarrow YZ}) = Q_2(a^{\downarrow YZ})$ and $m_2^{\downarrow Z}(a^{\downarrow Z}) = Q_2^{\downarrow Z}(a^{\downarrow Z})$. Therefore

$$\begin{aligned} m_1 \oplus m_2 \ominus m_2^{\downarrow Z}(a) = Q_1 \oplus Q_2 \ominus Q_2^{\downarrow Z}(a) &= Q_1(a^{\downarrow XZ}) \cdot Q_2(a^{\downarrow YZ}) / Q_2^{\downarrow Z}(a^{\downarrow Z}) \\ &= \frac{m_1(a^{\downarrow XZ}) \cdot m_2(a^{\downarrow YZ})}{m_2^{\downarrow Z}(a^{\downarrow Z})} = m_1 \triangleright m_2(a). \end{aligned}$$

□

Proposition 9. *Consider a Bayesian basic assignment m defined on $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$. If its marginal $m^{\downarrow Z}$ is* uniform *in the sense that for any two focal elements $z_1, z_2 \in \mathbb{Z}$ of $m^{\downarrow Z}$ it holds that $m^{\downarrow Z}(z_1) = m^{\downarrow Z}(z_2)$, then*

$$m^{\downarrow XZ} \oplus m^{\downarrow YZ} = m^{\downarrow XZ} \triangleright m^{\downarrow YZ}.$$

Proof. Since the proposition concerns only Bayesian basic assignments we need not care about other sets from $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$ than singletons, because for all "non-singletons" A

$$m^{\downarrow XZ} \oplus m^{\downarrow YZ}(A) = m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(A) = 0.$$

So, consider $a \in \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$. The Dempster's rule of combination yields

$$m^{\downarrow XZ} \oplus m^{\downarrow YZ}(a) = \frac{1}{K} \sum_{B\subseteq\mathbb{X}\times\mathbb{Z}, C\subseteq\mathbb{Y}\times\mathbb{Z}: \{a\}=B\bowtie C} m^{\downarrow XZ}(B) \cdot m^{\downarrow YZ}(C) = \frac{1}{K} m^{\downarrow XZ}(a^{\downarrow XZ}) \cdot m^{\downarrow YZ}(a^{\downarrow YZ}),$$

because among the singletons (i.e. focal elements of m) there cannot be other points whose join would yield point a. Now, we will use the fact that both $m^{\downarrow XZ} \oplus m^{\downarrow YZ}$ and $m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$ are normalized Bayesian basic assignments.

$$\begin{aligned} 1 &= \sum_{a\in\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}} m^{\downarrow XZ} \oplus m^{\downarrow YZ}(a) = \frac{1}{K} \sum_{a\in\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}} m^{\downarrow XZ}(a^{\downarrow XZ}) \cdot m^{\downarrow YZ}(a^{\downarrow YZ}) \\ &= \frac{1}{K} \sum_{a\in\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}} \left(m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(a)\right) \cdot m^{\downarrow Z}(a^{\downarrow Z}) = \frac{M}{K} \sum_{a\in\mathbb{X}\times\mathbb{Y}\times\mathbb{Z}} m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(a) = \frac{M}{K}, \end{aligned}$$

where $M = m^{\downarrow Z}(z)$ for all focal elements $z \in \mathbb{Z}$ of $m^{\downarrow Z}$. Thus we got $M = K$, and therefore for all $a \in \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$

$$m^{\downarrow XZ} \oplus m^{\downarrow YZ}(a) = \frac{m^{\downarrow XZ}(a^{\downarrow XZ}) \cdot m^{\downarrow YZ}(a^{\downarrow YZ})}{M} = m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(a),$$

which finishes the proof. □

Remark 3. The reader probably noticed that the statements of Propositions 8 and 9 are not surprising because for Bayesian basic assignment, which represent probability measures, both types of factorization coincide.

4.3 Example 2

Let X, Y, Z be binary variables with values in $\mathbb{X} = \{\xi, \bar{\xi}\}$, $\mathbb{Y} = \{\eta, \bar{\eta}\}$ and $\mathbb{Z} = \{\zeta, \bar{\zeta}\}$, and a basic assignment m with just two focal elements:

$$m(\{(\xi,\eta,\zeta)\}) = \tfrac{1}{2}, \quad m(\{(\xi,\eta,\zeta),(\xi,\eta,\bar{\zeta})\}) = \tfrac{1}{2}.$$

Let us compute $m^{\downarrow XZ} \oplus m^{\downarrow YZ}$ and $m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$.

For this we need the respective marginal basic assignments. Both of them have again two focal elements:

$$m^{\downarrow XZ}(\{(\xi,\zeta)\}) = \tfrac{1}{2}, \quad m^{\downarrow XZ}(\{(\xi,\zeta),(\xi,\bar{\zeta})\}) = \tfrac{1}{2}$$
$$m^{\downarrow YZ}(\{(\eta,\zeta)\}) = \tfrac{1}{2}, \quad m^{\downarrow YZ}(\{(\eta,\zeta),(\eta,\bar{\zeta})\}) = \tfrac{1}{2}$$

To compute $m^{\downarrow XZ} \oplus m^{\downarrow YZ}$, it is advantageous to find all sets $A \subseteq \mathbb{X} \times \mathbb{Y} \times \mathbb{Z}$ such that $A = B \bowtie C$, where B and C are focal elements of $m^{\downarrow XZ}$ and $m^{\downarrow YZ}$, respectively:

$$\begin{aligned}
&\{(\xi,\zeta)\} \bowtie \{(\eta,\zeta)\} = \{(\xi,\eta,\zeta)\},\\
&\{(\xi,\zeta)\} \bowtie \{(\eta,\zeta),(\eta,\bar{\zeta})\} = \{(\xi,\eta,\zeta)\},\\
&\{(\xi,\zeta),(\xi,\bar{\zeta})\} \bowtie \{(\eta,\zeta)\} = \{(\xi,\eta,\zeta)\},\\
&\{(\xi,\zeta),(\xi,\bar{\zeta})\} \bowtie \{(\eta,\zeta),(\eta,\bar{\zeta})\} = \{(\xi,\eta,\zeta),(\xi,\eta,\bar{\zeta})\}.
\end{aligned}$$

Employing this knowledge we get

$$\begin{aligned}
m^{\downarrow XZ} \oplus m^{\downarrow YZ}(\{(\xi,\eta,\zeta)\}) &= m^{\downarrow XZ}(\{(\xi,\zeta)\}) \cdot m^{\downarrow YZ}(\{(\eta,\zeta)\}) + m^{\downarrow XZ}(\{(\xi,\zeta)\}) \cdot m^{\downarrow YZ}(\{(\eta,\zeta),(\eta,\bar{\zeta})\})\\
&\quad + m^{\downarrow XZ}(\{(\xi,\zeta),(\xi,\bar{\zeta})\}) \cdot m^{\downarrow YZ}(\{(\eta,\zeta)\}) = \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{3}{4},
\end{aligned}$$

and

$$m^{\downarrow XZ} \oplus m^{\downarrow YZ}(\{(\xi,\eta,\zeta),(\xi,\eta,\bar{\zeta})\}) = m^{\downarrow XZ}(\{(\xi,\zeta),(\xi,\bar{\zeta})\}) \cdot m^{\downarrow YZ}(\{(\eta,\zeta),(\eta,\bar{\zeta})\}) = \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}.$$

As we said at the beginning of Section 4.2, computation of $m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$ is simpler. It is enough just to apply point [**a**] of Definition 3 (for $m^{\downarrow Z}$ see Table 2):

$$m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(\{(\xi,\eta,\zeta)\}) = \frac{m^{\downarrow XZ}(\{(\xi,\zeta)\}) \cdot m^{\downarrow YZ}(\{(\eta,\zeta)\})}{m^{\downarrow Z}(\{(\zeta)\})} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2},$$

$$m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(\{(\xi,\eta,\zeta),(\xi,\eta,\bar{\zeta})\}) = \frac{m^{\downarrow XZ}(\{(\xi,\zeta),(\xi,\bar{\zeta})\}) \cdot m^{\downarrow YZ}(\{(\eta,\zeta),(\eta,\bar{\zeta})\})}{m^{\downarrow Z}(\{(\zeta,\bar{\zeta})\})} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}.$$

It is obvious that for all the other sets $m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$ equal 0. In this way we got that though $m^{\downarrow XZ} \oplus m^{\downarrow YZ}$ and $m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$ have the same focal elements

$$m^{\downarrow XZ} \oplus m^{\downarrow YZ}(\{(\xi,\eta,\zeta)\}) = \tfrac{3}{4}, \qquad m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(\{(\xi,\eta,\zeta)\}) = \tfrac{1}{2},$$

$$m^{\downarrow XZ} \oplus m^{\downarrow YZ}(\{(\xi,\eta,\zeta),(\xi,\eta,\bar{\zeta})\}) = \tfrac{1}{4}, \quad m^{\downarrow XZ} \triangleright m^{\downarrow YZ}(\{(\xi,\eta,\zeta),(\xi,\eta,\bar{\zeta})\}) = \tfrac{1}{2},$$

they differ from each other in the values assigned to them. Thus we see that basic assignment m factorizes in the sense that $m = m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$.

Table 2. Marginal basic assignment $m^{\downarrow Z}$ and its commonality function

A	$m^{\downarrow Z}$	$Q^{\downarrow Z}$
ζ	$\frac{1}{2}$	1
$\bar{\zeta}$	0	$\frac{1}{2}$
$\zeta, \bar{\zeta}$	$\frac{1}{2}$	$\frac{1}{2}$

Let us finish this example by computing $m^{\downarrow XZ} \oplus m^{\downarrow YZ} \ominus m^{\downarrow Z}$. To do it, we have to find a basic assignment corresponding to commonality function $Q^{\downarrow XZ} \oplus Q^{\downarrow YZ} \ominus Q^{\downarrow Z}$. The necessary computations are visible from Table 3. Realize that the given set functions are generally defined for 255 sets, however, for all of them, but for the three presented in Table 3, all values of the presented basic assignments and commonality functions equal 0.

Table 3. Combined basic assignments and their commonality functions

A	$m^{\downarrow XZ} \oplus m^{\downarrow YZ}$	$Q^{\downarrow XZ} \oplus Q^{\downarrow YZ}$	$Q^{\downarrow XZ} \oplus Q^{\downarrow YZ} \ominus Q^{\downarrow Z}$	$m^{\downarrow XZ} \oplus m^{\downarrow YZ} \ominus m^{\downarrow Z}$
(ξ, η, ζ)	$\frac{3}{4}$	1	$1/1 = 1$	$\frac{1}{2}$
$(\xi, \eta, \bar{\zeta})$	0	$\frac{1}{4}$	$\frac{1}{4}/\frac{1}{2} = \frac{1}{2}$	0
$(\xi, \eta, \zeta), (\xi, \eta, \bar{\zeta})$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}/\frac{1}{2} = \frac{1}{2}$	$\frac{1}{2}$

5 Summary and Conclusions

The goal of the paper was to discuss relationship between two ways of factorization within Dempster-Shafer theory of evidence. Necessity of factorization of multidimensional basic assignments (or an equivalent representation) follows from a high algorithmic complexity of all the computational procedures connected with an inference within the Dempster-Shafer theory framework. With respect to this, let us say that in this paper we spoke only about factorization of basic assignments and commonality functions. Though it is used in the title, we did not studied a factorization of the respective belief of plausibility functions.

The types of factorization we spoke about are closely connected with the two types of operators defined for basic assignment combination: the Dempster's rule of combination and the operator of composition. These two operators were designed for different purposes and both of them meet the basic requirement expected from factorization, namely they enable constructing multidimensional models from a system of low-dimensional ones.

The paper contains just a preliminary ideas and gives answers only to very simple questions. So there are many more that remain to be answered. For example:

- Is it possible to characterize situations when $m_1 \oplus m_2 \ominus m_2^{\downarrow Z} = m_1 \triangleright m_2$?
- Is it possible to compute locally $m(X, Z|Y)$ for $m = m^{\downarrow XZ} \triangleright m^{\downarrow YZ}$?
- When $\left(m_1 \oplus m_2 \ominus m_2^{\downarrow Z}\right)^{\downarrow XZ} = m_1$?

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