A three-person cooperative game formulation of the world oil market

Prakash P. Shenoy

The School of Business, The University of Kansas, Lawrence, Kansas 66045, USA
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The world oil market is modelled as a three-person cooperative game in characteristic function form with and without side payments. The three players are the oil importing countries, the group in OPEC led by Iran and the group in OPEC led by Saudi Arabia. Various solutions of the games are studied such as the core, the Shapley value, the bargaining set and the nucleolus.

Introduction

In the winter of 1973, some major oil exporting countries joined together to declare an embargo on oil exports to some of the western countries for political reasons. Elated by their success and the realization that they controlled a major share of the oil exports, they subsequently raised the price of oil four-fold and cut back production, thereby obtaining, in the face of an almost inelastic demand, increased revenues.

The major oil importing countries have been trying to work out an optimal policy designed to obtain their energy needs at lowest possible prices. One of the strategies considered by these countries is to attempt to split the oil cartel by bilateral dealings or by trying to play one member off against another. This paper analyses the feasibility of such a strategy and its cost in financial terms using the theory of n-person cooperative games.

In Shenoy,[1] the world oil market is modelled as a two-person non-zero-sum game with the oil importing countries denoted by OPIC as one player and the oil exporting countries denoted by OPEC as the second player. In this paper, we divide OPEC into two groups: one led by Saudi Arabia (SA) and the other led by Iran (IR). Despite many common characteristics, each group displays different national attributes and long-term commercial interests. IR, with a larger population, relatively small petroleum reserves, aggressive plans for economic development and military build-up, can use all the revenue available through major price increases. SA on the other hand, has a very small population and hence little capital absorption capability, large petroleum reserves and enormous financial reserves. In a period of rapid inflation, SA would prefer to have the oil in the ground rather than increase production. Also, SA would prefer to keep prices below the substitution threshold for new energy sources because of the fear that a flood of new energy will drive the price downward substantially in advance of the time when SA's petroleum reserves are exhausted. Although huge time lags of seven years or more are involved in energy substitution, SA fears the impact of potential large scale economies in coal liquefication and other related techniques and the possibility of a significant breakthrough in terms of the leaning curve, all of which would help to bring down the future price of energy. IR, facing a much shorter time horizon for the exhaustion of its energy reserves, can push the price of crude oil very high without much fear of the consequences from accelerating new discoveries and the innovation of new sources of energy.[2]

The models

The world oil market is modelled as a three-person cooperative game in characteristic function form with and without side payments. The characteristic function form of the game is chosen because it focuses on the bargaining process and allocation of payoffs among the players.

Player 1 called OPIC represents all the oil importing countries. Here were assume that all the major oil importing countries have formed a cartel and bargain as one unit. Player 2 called IR and player 3 called SA represent the two groups in the OPEC cartel that have between them all the oil exported to OPIC, who we assume is the sole market for the oil exports.

We shall assume that OPIC needs a total of I million barrels of oil daily (mmbd) assuming consumption required for a maximum growth of their economy. A part of this requirement can be met by domestic production of oil. By a large investment, the domestic production of oil can be increased by finding new sources, or simply working the existing wells harder using improved technology. Alternatively, the demand for oil can partly be satisfied by other fuels such as coal, nuclear fussion, shale oil and other new sources that could be developed by a large investment in research and development. Furthermore, the consumption of oil could be reduced by voluntary or mandatory methods such as rationing the supply of oil, and energy tax, etc. This may, however, resuit in losses in the nation's economy.

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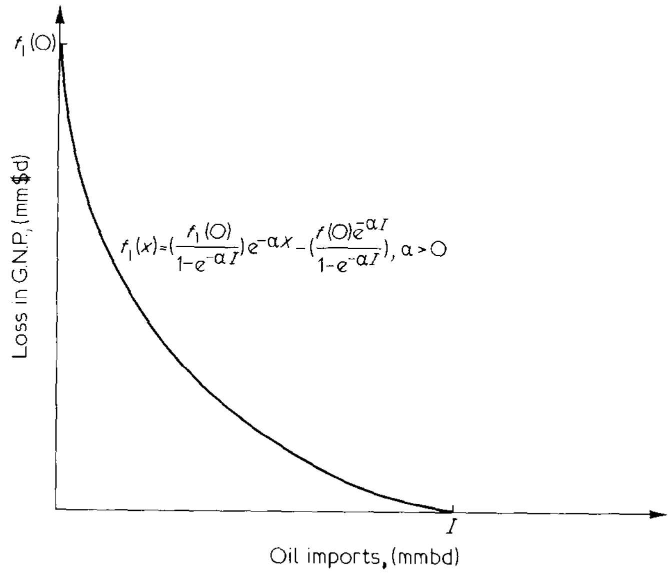

Figure 1 Sketch of possible nature of function f_1

In short, the strategy for OPIC is to decide the quantity of oil imports from SA and IR. More formally, the strategy space of OPIC is denoted by:

$$\Sigma_1 = \{(x_2, x_3) \in E^2 \quad 0 \leqslant x_2 + x_3 \leqslant I\}$$

Associated with a strategy $(x_2, x_3) \in \Sigma_1$ is a monetary cost to OPIC, denoted by $f_1(x_1)$ where $x_1 = x_2 + x_3$, for restricting its imports to x_1 mmbd. $f_1(x_1)$ does not include the cost of imports. A sketch of a method of computing $f_1(x_1)$ is as follows.

Let $h(y)$ denote the total cost in million dollars daily (mm$d) to ensure that domestic production of oil is at least y mmbd. Let $g(z)$ denote the loss in mm$d in OPIC's GNP* if the total oil (energy) consumption is restricted to z mmbd. Then we have:

$$f_1(x_1) = \min_{0 \leqslant y \leqslant I - x_1} [h(y) + g(y + x_1)]$$

We will assume that $f_1(x_1)$ is a nonincreasing, positive, real-valued function defined on the strategy space Σ_1 of OPIC. Several studies havebeen made to determine the function $f_1(x_1)$ for the case of the USA alone. See Shenoy[1] for more details. A sketch of the possible nature of $f_1(x_1)$ is indicated in *Figure 1*.

Let C_2 and C_3 denote the production capacities (in million barrels of oil daily) of IR and SA respectively. Let e_2 and e_3 denote the extraction cost in dollars per barrel of oil ($/b) for IR and SA respectively. Also let M_2 and M_3 denote the capital investment in million dollars daily (mm$d), necessary to achieve a maximum growth rate for IR's and SA's economy. For $0 \leqslant y \leqslant M_2$ and $0 \leqslant z \leqslant M_3$, let $f_2(y)$ and $f_3(z)$ denote the losses in mm$d to IR's and SA's economies if capital investment is restricted to y and z mm$d respectively. Any capital in excess of M_2 and M_3 is available as capital reserves. We will assume that f_2 and f_3 are nonincreasing, real-valued functions defined on the real interval $[0, \infty)$. A sketch of the possible nature of these functions is shown in *Figure 2*.

* Gross National Product. Other indicators of a nation's economy can also be used

The strategy for both IR and SA is to decide on the price of oil exported. Let:

$$\Sigma = \{(x_2, x_3; p_2; p_3): \ 0 \leqslant x_2 + x_3 \leqslant I, e_2 \leqslant p_2 < \infty, e_3 \leqslant p_3 < \infty, 0 \leqslant x_2 \leqslant C_2, 0 \leqslant x_3 \leqslant C_3\}$$

denote the set of all possible outcomes.

For each outcome in Σ, there results a monetary payoff to each player. Let A_i: $\Sigma \to E^1$ denote the (monetary) payoff function of player i ($i = 1, 2, 3$). Then we define:

$$A_1(x_2, x_3; p_2; p_3) = -f_1(x_2 + x_3) - p_2 x_2 - p_3 x_3$$
$$A_2(x_2, x_3; p_2; p_3) = -f_2((p_2 - e_2)x_2)$$
$$A_3(x_2, x_3; p_2; p_3) = -f_3((p_3 - e_3)x_3)$$

Before defining the characteristic functions of the side payment and the non-side payment game, we will make the following assumptions regarding the parameters of the problem.

$C_2 < C_3 < I < C_2 + C_3$	A1
$f_3(0) < f_2(0) < f_1(0)$	A2
$M_3 < M_2$	A3
$e_2 < \lvert f_1'(x_1) \rvert$ for each $0 \leqslant x_1 \leqslant I$	A4
$e_3 < \lvert f_1'(x_1) \rvert$ for each $0 \leqslant x_1 \leqslant I$	A5
$\lvert f_2'(x_2) \rvert > 1$ for each $0 \leqslant x_2 \leqslant M_2$	A6
$\lvert f_3'(x_3) \rvert > 1$ for each $0 \leqslant x_3 \leqslant M_3$	A7

Assumptions A1–A7 represent the realities of the situation being modelled.

Side payment model

In this section, we will assume that unrestricted side payments are allowed. We will use the von Neumann–Morgenstern[3] model of the characteristic function. This is derived by considering the maximum each coalition can guarantee itself under any circumstances. Also, we assume that utility is linear in money.

Let $N = \{1, 2, 3\}$ denote the set of players; 2^N, the set of all subsets of N called coalitions; and v: $2^N \to E^1$, the characteristic function which is defined as follows:

$$v(\{\emptyset\}) = 0, \ v(\{1\}) = -f_1(0)$$
$$v(\{2\}) = -f_2(0), \ v(\{3\}) = -f_3(0)$$
$$v(\{1, 2\}) = \max_{\substack{0 \leqslant x_2 \leqslant C_2 \\ e_2 \leqslant p_2 < \infty}} \{-f_1(x_2) - p_2 x_2 - f_2((p_2 - e_2)x_2)\}$$
$$v(\{2, 3\}) = -f_2(0) - f_3(0)$$
$$v(\{1, 3\}) = \max_{\substack{0 \leqslant x_3 \leqslant C_3 \\ e_3 \leqslant p_3 < \infty}} \{-f_1(x_3) - p_3 x_3 - f_3((p_3 - e_3)x_3\}$$

and

$$v(\{1, 2, 3\}) = \max_{(x_2, x_3; p_2; p_3) \in \Sigma} \{-f_1(x_2 + x_3) - p_2 x_2 - p_3 x_3 - f_2((p_2 - e_2)x_2) - f_3((p_3 - e_3)x_3)\}$$

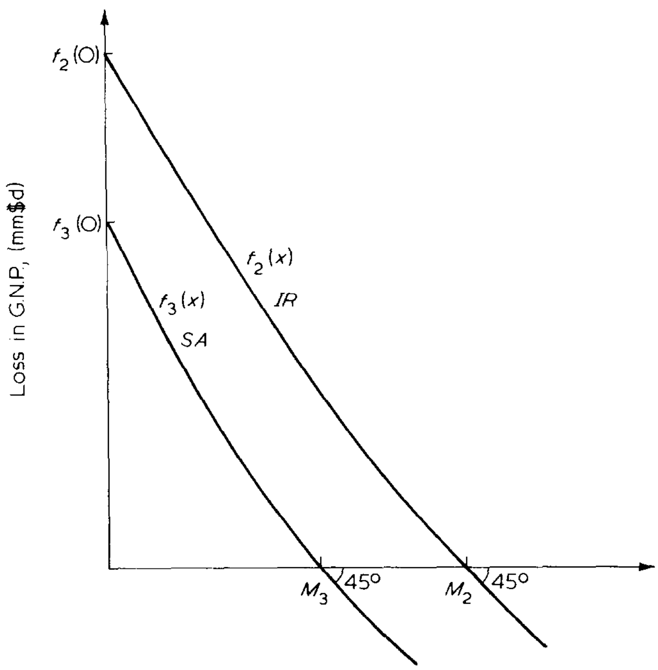

Figure 2 Sketch of possible nature of functions f_2 and f_3

We shall now determine the relative magnitudes of the values of the characteristic function. We have*:

$$v(12) = \max_{\substack{0 \leqslant x_2 \leqslant C_2 \\ e_2 \leqslant p_2 < \infty}} \{-f_1(x_2) - p_2 x_2 - f_2((p_2 - e_2)x_2)\}$$

Clearly, by assumptions A4 and A6, the maximum in the above expression is achieved at $x_2 = C_2, p_2 = e_2 + (M_2/C_2)$, and so we obtain:

$$v(12) = -f_1(C_2) - e_2C_2 - M_2 - f_2(M_2)$$
$$= -f_1(C_2) - e_2C_2 - M_2$$

Similarly, we obtain:

$$v(13) = -f_1(C_3) - e_3C_3 - M_3$$

and

$$v(123) = -f_1(I) - K - M_1 - M_2$$
$$= -K - M_1 - M_2$$

where $K = \min\{e_2C_2 + e_3(I - C_2), e_3C_3 + e_2(I - C_3)\}$.

In 0-normalized form, the characteristic function is as follows:

$$v(\emptyset) = v(1) = v(2) = v(3) = 0$$
$$v(12) = f_1(0) + f_2(0) - f_1(C_2) - e_2C_2 - M_2$$
$$v(13) = f_1(0) + f_3(0) - f_1(C_3) - e_3C_3 - M_3$$
$$v(23) = 0$$

and

$$v(123) = f_1(0) + f_2(0) + f_3(0) - K - M_2 - M_3$$

we make two additional assumptions as follows:

$$f_3(0) - f_1(C_3) - e_3C_3 - M_3 > f_2(0) - f_1(C_2) - e_2C_2 - M_2 \qquad \text{A8}$$

$$f_2(0) - K - M_2 > -f_1(C_3) - e_3C_3 \qquad \text{A9}$$

* To condense notation, we shall drop the parenthesis around the players in a coalition and denote, for example $v(\{1, 2\})$ by $v(12)$

These assumptions, again, reflect the realities of the situation. So we have the following relation:

$$0 < v(12) < v(13) < v(123)$$

Non-side payment model

We consider cooperative games without side payments because it approximates the real life situation more closely than games with side payments. Although side payments are legal, utility is usually nonlinear in money and this results in a situation not covered by the von Neumann–Morgenstern theory.[4,5]

Before defining the characteristic function of the non-side payment game, we will define the utility function of each of the players. Each monetary payoff has a particular utility to each player. Let u_i: $A_i(\Sigma) \rightarrow [0, 1]$ denote the utility function of player i ($i = 1, 2, 3$). Define:

$$u_1(z) = \frac{f_1(0) + z}{f_1(0) - K} \quad \text{if } -f_1(0) \leqslant z \leqslant -K$$

$$u_2(z) = \frac{f_2(0) + z}{f_2(0)} \quad \text{if } -f_2(0) \leqslant z \leqslant 0$$

$$u_3(z) = \frac{f_3(0) + z}{f_3(0)} \quad \text{if } -f_3(0) \leqslant z \leqslant 0$$

We can now define the characteristic function of the non-side payment game. Denote E^S the subspace of E^3 spanned by the axes belonging to the players in a subset $S \subset N$. The characteristic function V: $2^N \rightarrow E^3$ associates with each coalition $S \subset N$, a subset $V(S)$ of E^S. Intuitively, $V(S)$ represents the set of payoff vectors that the coalition S can guarantee itself. Let E^3_+ denote the positive orthant of E^3. Also let $\mathrm{Conv}\{a_1, \ldots, a_p\}$ denote the convex hull of the vectors in $\{a_1, \ldots, a_p\}$.

We define V as follows:

$$V(\emptyset) = E^3_+$$

$$V(1) = (u_1(-f_1(0)), 0, 0) - E^3_+$$
$$= (0, 0, 0) - E^3_+$$

$$V(2) = (0, u_2(-f_2(0)), 0) - E^3_+$$
$$= (0, 0, 0) - E^3_+$$

$$V(3) = (0, 0, u_3(-f_3(0))) - E^3$$
$$= (0, 0, 0) - E^3_+$$

$$V(12) = \mathrm{Conv}\{(u_1(-f_1(C_2) - e_2C_2 - M_2), u_2(-f_2(M_2)), 0)$$
$$(u_1(-f_1(C_2) - e_2C_2), u_2(-f_2(0)), 0)\} - E^3_+$$
$$= \mathrm{Conv}\{(u_1(-f_1(C_2) - e_2C_2 - M_2), 1, 0)$$
$$(u_1(-f_1(C_2) - e_2C_2), 0, 0\} - E^3_+$$

$$V(13) = \mathrm{Conv}\{(u_1(-f_1(C_3) - e_3C_3 - M_3), 0, u_3(-f_3(M_3)))$$
$$(u_1(-f_1(C_3) - e_3C_3), 0, u_3(-f_3(0)))\} - E^3_+$$
$$= \mathrm{Conv}\{(u_1(-f_1(C_3) - e_3C_3 - M_3), 0, 1),$$
$$(u_1(-f_1(C_3) - e_3C_3), 0, 0)\} - E^3_+$$

$$V(23) = (0, u_2(-f_2(0)), u_3(-f_3(0))) - E^3_+$$
$$= (0, 0, 0) - E^3_+$$

$$V(123) = \mathrm{Conv}\{(u_1(-f_1(I) - K), u_2(-f_2(0)), u_3(-f_3(0)))$$
$$(u_1(-f_1(I) - K - M_2), u_2(-f_2(M_2)), u_3(-f_3(0)))$$
$$(u_1(-f_1(I) - K - M_3), u_2(-f_2(0)), u_3(-f_3(M_3)))$$

$$(u_1(-f_1(I)-K-M_2-M_3), u_2(-f_2(M_2)),$$
$$u_3(-f_3(M_3)))\} - E^3_+$$
$$= \text{Conv}\{(1,0,0), (u_1(-f_1(I)-K-M_2), 1, 0)$$
$$(u_1(-f_1(I)-K-M_3), 0, 1)$$
$$(u_1(-f_1(I)-K-M_2-M_3), 1, 1)\} - E^3_+$$

Here we assume that:

$$-f_1(I)-K-M_2-M_3 > -f_1(0) \qquad \text{A10}$$

See *Figures 3* and *4* for a geometrical representation of the above game. This completes the formulation of the world oil market as a non-side payment game. In the subsequent sections, we study the solutions of the side payment and the non-side payment game.

Solutions of the side payment game

There are many solution concepts for n-person cooperative games with side payments. Each solution has its own intuitive justification. In this section we will study the core, the Shapley value, the bargaining set, the nucleolus, and the normalized nucleolus.

Let us denote the characteristic function defined in the section on the side payment model as follows:

$$v(123) = \gamma \quad v(13) = \beta \quad v(12) = \alpha$$

where $\gamma > \beta > \alpha$.

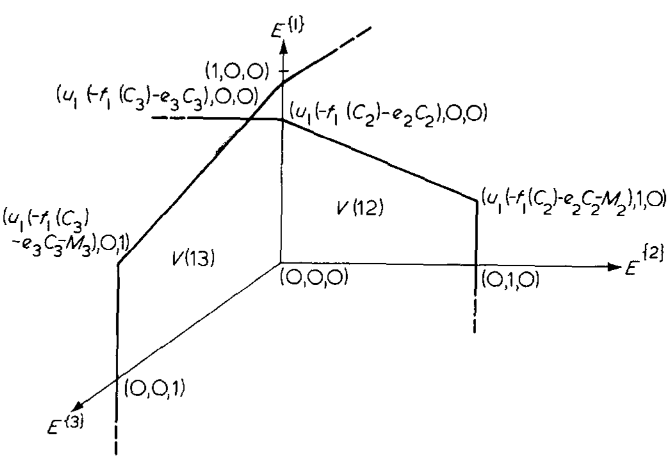

Figure 3 Geometrical representation of $V(12)$ and $V(13)$

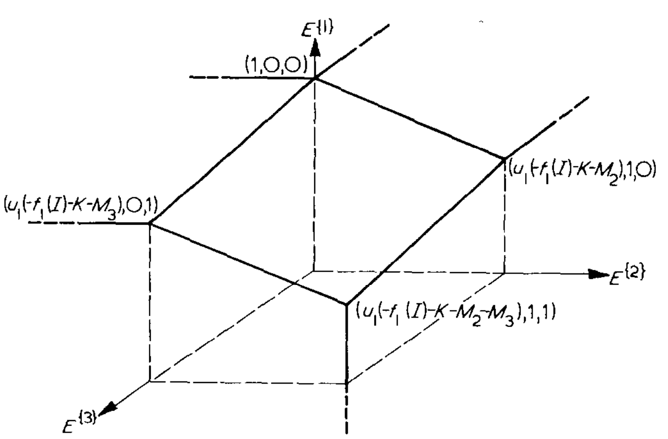

Figure 4 Geometrical representation of $V(123)$

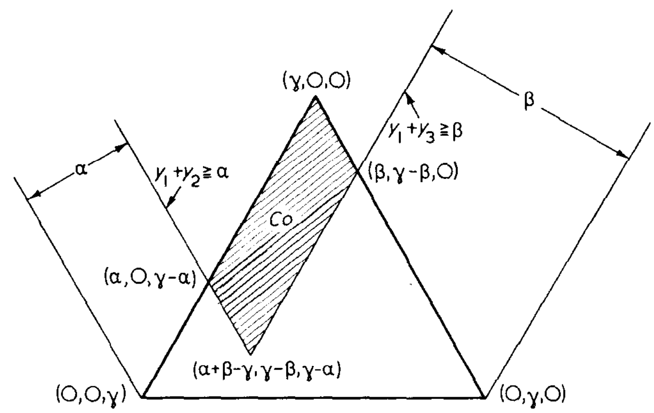

Figure 5 Geometrical representation of core (in barycentric coordinates), Case (i)

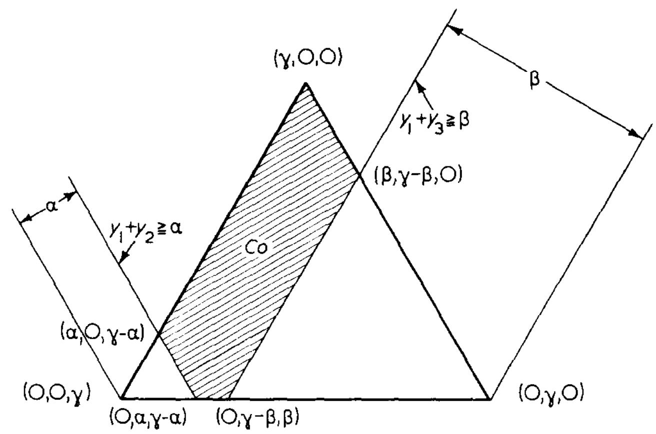

Figure 6 Geometrical representation of core (in barycentric coordinates), Case (ii)

Core

The core of a game with side payments was first studied by Gillies[6] and Shapley. An imputation in this game is any vector (y_1, y_2, y_3) such that:

$$y_1 \geqslant v(1),\ y_2 \geqslant v(2),\ y_3 \geqslant v(3) \text{ (individual rationality)}$$

and

$$y_1 + y_2 + y_3 = v(123) \text{ (Pareto optimality)}$$

y_1, y_2 and y_3 represent payoffs to players 1, 2, and 3 respectively. The core of our game consists of those imputations (if any) which satisfy the following relations.

$$y_1 + y_2 \geqslant \alpha \quad y_1 + y_3 \geqslant \beta \quad y_2 + y_3 \geqslant 0$$

Let Co denote the core of our game. Then it is given as follows:

Case (i) $\beta < \gamma \leqslant \alpha + \beta$ (see *Figure 5*)

$$Co = \text{Conv}\{(\gamma,0,0), (\beta, \gamma-\beta, 0), (\alpha+\beta-\gamma, \gamma-\beta, \gamma-\alpha), (\alpha, 0, \gamma-\alpha)\}$$

Case (ii) $\gamma > \alpha + \beta$ (see *Figure 6*)

$$Co = \text{Conv}\{(\gamma,0,0), (\beta, \gamma-\beta, 0), (0, \gamma-\beta, \beta), (0, \alpha, \gamma-\alpha), (\alpha, 0, \gamma-\alpha)\}$$

The outcomes in the core have to be interpreted carefully. The core as defined above assumes that IR and SA (players 2 and 3) are acting independently without any collusion. Also we assume that all the oil consumers are acting together as one player. (These assumptions are not based on reality but describe a scenario where OPEC splits up into two and the oil consuming countries form a cartel). In this situation, we have a market with one buyer (OPIC) and two sellers (IR and SA). As would be intuitively expected, OPIC is at an advantage since it can play one seller off against another. The outcomes in the core reflect this. Also the core indicates that SA is in a relatively better position compared to IR. This is also to be expected as SA has more oil than IR and also has a lesser need for revenue compared to IR. The core consists of many outcomes and does not distinguish any particular imputation as more likely than others.

Shapley value

The rationale for the Shapley[7] value is in terms of the bargaining power which each player imagines he possesses. This power (as estimated by the player in question) is based on what his joining each coalition contributes to that coalition.

For a three-person game, the Shapley value, denoted by (ϕ_1, ϕ_2, ϕ_3) is as follows:

$$\phi_1 = 1/3(v(123) - v(23)) + 1/6(v(12) - v(2)) + 1/6(v(13) - v(3)) + 1/3(v(1) - v(\emptyset))$$

$$\phi_2 = 1/3(v(123) - v(13)) + 1/6(v(12)) - v(1)) + 1/6(v(23) - v(3)) + 1/3(v(2) - v(\emptyset))$$

$$\phi_3 = 1/3(v(123) - v(12)) + 1/6(v(13) - v(1)) + 1/6(v(23) - v(2)) + 1/3(v(3) - v(\emptyset))$$

Substituting the values of the characteristic function in the above expressions, we obtain:

$$\phi_1 = (2\gamma + \alpha + \beta)/6$$

$$\phi_2 = (2\gamma + \alpha - 2\beta)/6$$

$$\phi_3 = (2\gamma - 2\alpha + \beta)/6$$

Note that $\phi_1 + \phi_2 + \phi_3 = \gamma$. Also since $\gamma > \beta > \alpha$, we have:

$$\phi_1 > \phi_3 > \phi_2$$

The Shapley value also indicates that OPIC has an advantage over IR and SA and SA has an edge over IR. The Shapley value besides determining a unique allocation of the payoff solely by the characteristic function of the game has a certain equity principle built into it. This solution might therefore be a strong contender for the status of a 'normative' solution, i.e., one which 'rational players' ought to accept. Its weakness is that it derives entirely from the characteristic function of the game and not from what is 'beneath' the characteristic function, i.e., the strategic structure of the game itself rather than the bargaining positions of the players in the process of coalition formation.

Bargaining set $M_1^{(i)}$

The bargaining set was first introduced by Aumann and Maschler[8] (A–M). The A–M bargaining set was developed to attack the following general question. If the players in a cooperative n-person game have decided upon a specific coalition structure, how then will they distribute the values of the various coalitions among themselves in such a way that some stability requirements will be satisfied (cf. Davis and Maschler[9] (p. 39). These stability requirements are based on the idea that a 'stable' payoff configuration should offer some security in the sense that each 'objection' could be met by a 'counterobjection'. Several kinds of bargaining sets were defined. One of these denoted by $M_1^{(i)}$ was shown by Peleg[10] to be nonempty for each partitioning of the players into a coalition structure.

The bargaining set $M_1^{(i)}$ for our game is given by:

$$M_1^{(i)}(P) = \begin{cases} (0, 0, 0) & \text{if } P = (1)(2)(3)^* \\ (\alpha, 0, 0) & \text{if } P = (12)(3) \\ (\alpha \leq y_1 \leq \beta, 0, \beta - y_1)^\dagger & \text{if } P = (13)(2) \\ (0, 0, 0) & \text{if } P = (1)(23) \\ Co & \text{if } P = (123) \end{cases}$$

The bargaining set corresponding to the grand coalition coincides with the core. The bargaining set also indicates that when OPIC and IR are in a coalition against SA, IR has no bargaining power at all vis-a-vis OPIC. An observation of all the outcomes in the bargaining set reveals that it is in the mutual interest of all the players to form the grand coalition (consisting of all three players).

Nucleolus and normalized-nucleolus

The nucleolus, ν, was defined by Schmeidler.[11] Let $y = (y_1, y_2, y_3)$ be an imputation. Then the excess of coalition R with respect to imputation y is:

$$e_R(y) = v(R) - \sum_{i \in R} y_i$$

The excess of coalition R with respect to imputation y is a measure of coalition R's 'complaint' against imputation y. The nucleolus is that imputation which minimizes the 'loudest complaint'. (In case of a tie in the largest complaint, the next largest excesses are compared etc.) The nucleolus consists of a unique imputation in the bargaining set $M_1^{(i)}$ and the core if the latter is nonempty.

The normalized-nucleolus (n-nucleolus) μ suggested by Lucas and studied by Grotte[12] is defined in the same manner as the nucleolus except that excesses $e_R(y)$ are replaced by normalized-excesses:

$$e_R^{\mu}(y) = \frac{e_R(y)}{|R|}$$

where $|R|$ denotes the cardinality of coalition R.

The nucleolus ν for our game is given as follows:

Case (i) $\gamma > 3\beta$

$$\nu = (\gamma/3, \gamma/3, \gamma/3)$$

Case (ii) $\beta + 2\alpha \leq \gamma \leq 3\beta$

$$\nu = ((\gamma + \beta)/4, (\gamma - \beta)/2, (\gamma + \beta)/4)$$

Case (iii) $\alpha + \beta \leq \gamma \leq \beta + 2\alpha$

$$\nu = ((\alpha + \beta)/2, (\gamma - \beta)/2(\gamma - \alpha)/2$$

Case (iv) $\beta \leq \gamma \leq \alpha + \beta$

$$\nu = ((\alpha + \beta)/2, (\gamma - \beta)/2, (\gamma - \alpha)/2)$$

The n-nucleolus μ for our game is given (in all cases) by:

$$\mu = ((2\gamma + \beta + 3\alpha)/6, (\gamma - \beta)/3, (2\gamma + \beta - 3\alpha)/6)$$

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* For convenience of notation, the partition $\{1\}, \{2\}, \{3\}$ is denoted by (1) (2) (3), etc.

† Denotes the set $\{(y_1, 0, \beta - y_1): \alpha \leq y_1 \leq \beta\}$

If we denote $\nu = (\nu_1, \nu_2, \nu_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ then in all cases we have:

$$\nu_1 \geqslant \nu_3 \geqslant \nu_2 \quad \text{and} \quad \mu_1 \geqslant \mu_3 \geqslant \mu_2$$

Solutions of the non-side payment game

In this section, we study the core and the bargaining set of the non-side payment game defined earlier in the paper.

Core

The core of a game without side payments has been studied by Aumann,[13] Billera[14,15] and Scarf.[16] A vector of utility levels is suggested which is feasible for all the players acting collectively and an arbitrary coalition is examined to see whether it can provide higher utility levels for all of its members. If this is possible, the utility vector which was originally suggested is said to be dominated by the coalition. The core of the n-person game consists of those utility vectors which are feasible for the entire group of players and which can be dominated by no coalition.

For our game, the core C is given as follows:

$$C = \mathrm{Conv}\{(1,0,0), (u_1(-f_1(C_3) - e_3C_3), a_2, 0), (u_1(-f_1(C_3) - e_3C_3 - M_3), a_2', 1), (u_1(-f_1(I) - K - M_3), 0, 1)\}$$

where:

$$a_2 = \frac{1 - u_1(-f_1(C_3) - e_3C_3)}{1 - u_1(-f_1(I) - K - M_2)}$$

and

$$a_2' = \frac{u_1(-f_1(C_3) - e_3C_3 - M_3) - u_1(-f_1(I) - K - M_3)}{u_1(-f_1(I) - K - M_2 - M_3) - u_1(-f_1(I) - K - M_3)}$$

(see *Figure 7*). The core again exhibits the advantage of OPIC over SA and IR and the advantage of SA over IR.

Bargaining set

The A–M bargaining set $M_1^{(i)}$ was generalized by Peleg[17] to games without side payments. However, he showed that it may be empty for some games. Billera[18] proposed another bargaing set $\tilde{M}_1^{(i)}$ based on the following simple principle. A payoff vector y is said to belong to the bargaining set $\tilde{M}_1^{(i)}$ if whenever player k has a justified objection (i.e., an objection that has no counterobjection) against player l at y, then there exists a chain of justified objections all at y leading from player l to player k (via other players). Asscher[19] proved that $\tilde{M}_1^{(i)}$ is never empty for games without side payments.

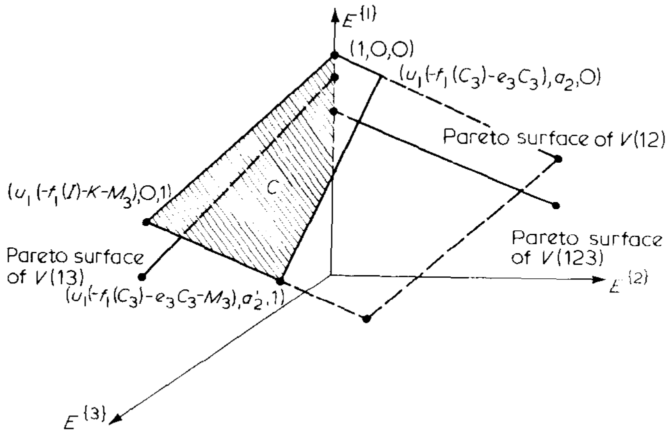

Figure 7 Geometrical representation of core of non-side payment game

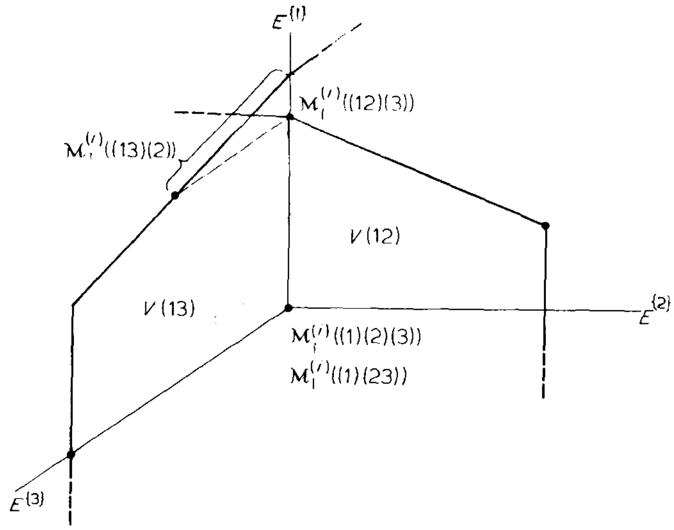

Figure 8 Bargaining set $M_1^{(i)}$ of non-side payment game

For our games we have $M_1^{(i)} = \tilde{M}_1^{(i)}$ and it is given as follows (see *Figure 8*):

$$M_1^{(i)} = \begin{cases} (0,0,0) & \text{if } P = (1)(2)(3) \\ (u_1(-f_1(C_2) - e_2C_3), 0, 0) & \text{if } P = (12)(3) \\ (0,0,0) & \text{if } P = (1)(23) \\ \mathrm{Conv}\{(u_1(-f_1(C_3) - e_2C_3), 0, 0) & \text{if } P = (13)(2) \\ \quad (u_1(-f_1(C_2) - e_2C_2), 0, a_3)\} & \\ C & \text{if } P = (123) \end{cases}$$

where:

$$a_3 = \frac{u_1(-f_1(C_3) - e_3C_3) - u_1(-f_1(C_2) - e_2C_2)}{u_1(-f_1(C_3) - e_3C_3) - u_1(-f_1(C_3) - e_3C_3 - M_3)}$$

As in the side payment case, the bargaining set for the grand coalition coincides with the core as determined in the previous section. Also it is observed that it is in the mutual interest of all the players to form the grand coalition.

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